

**Automorphic forms, Stark units  
and Heegner points**

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# Part I

## Introduction

### 1 Unit groups and Mordell-Weil groups

Let  $K$  be a number field. By Dirichlet's theorem,  $\mathcal{O}_K^\times$  is an abelian group of finite rank. *In what situations are there systematic ways for constructing (nontorsion) elements of  $\mathcal{O}_K^\times$ ?*

Let  $E/K$  be an elliptic curve. Then by the Mordell-Weil theorem,  $E(K)$  is an abelian group of finite rank. *In what situations are there systematic ways for constructing (nontorsion) elements of  $E(K)$ ?*

These two problems are related and should be considered together.

Let  $\zeta_K(s)$  be the Dedekind zeta function associated to  $K$  and set  $r = r_1 + r_2 - 1 = \text{rank } \mathcal{O}_K^\times$ . Then  $\zeta_K(s)$  is holomorphic at  $s = 0$  and has Taylor expansion of the form

$$\zeta_K(s) = -\frac{h_K R_K}{|(\mathcal{O}_K^\times)_{\text{tors}}|} s^r + O(s^{r+1}).$$

Let  $L(E/K, s)$  be the  $L$ -function of  $E/K$  and let  $r = \text{rank}(E/K)$ . It is conjectured that  $L(E/K, s)$  has analytic continuation to  $\mathbb{C}$  and that its Taylor expansion about  $s = 1$  has the form

$$L(E/K, s) = \frac{|\mathfrak{III}(E/K)| R_{E/K}}{|E(K)_{\text{tors}}|} (\Omega \prod c_p) (s-1)^r + O((s-1)^{r+1}).$$

These formulas possess a striking formal similarity. In fact, they are special cases of an Überconjecture formulated by Beilinson, Bloch, and Deligne.

*In these formulas,  $\mathcal{O}_K^\times$  and  $E(K)$  play analogous roles.*

Techniques we will discuss for constructing units have counterparts for constructing rational points and vice versa.

Automorphic forms (Eisenstein series, classical modular forms, Hilbert modular forms, automorphic forms for unit groups in quaternion algebras...) are the key tools in all the constructions we have.

The constructions and formulas that are not based directly on CM theory are *conjectural*. It seems that in many cases we can give local, analytic (archimedean and nonarchimedean!) formulas for objects that should be global, but I *no idea* how to prove their rationality.

Is there a “generalized CM theory” lurking in the background?

But before I get to that, I would like to set the stage by discussing a few aspects of classical CM theory.

## 2 “Cyclotomic CM theory”

**Theorem.** (Kronecker-Weber)

$$\mathbb{Q}^{\text{ab}} = \mathbb{Q}(\{e^{2\pi i/n} : n \geq 1\})$$

This follows immediately from class field theory for  $\mathbb{Q}$ : Let  $\mathbb{Q}[c]$  be the ray class field of  $\mathbb{Q}$  of conductor  $c$ . Then

$$\mathbb{Q}[c] = \mathbb{Q}(e^{2\pi i/c}).$$

Existence theorem: Every abelian number field has a conductor.

The ray class field of conductor  $c$  is the biggest number field of conductor  $c$ .

**Hilbert’s 12th problem.** Generalize Kronecker-Weber to arbitrary number fields  $K$ , i.e., show how to generate  $K^{\text{ab}}$  using special values of meromorphic functions.

### 3 CM theory

Let  $K$  be an imaginary quadratic field,  $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$ . Let  $\mathcal{O}_c \subset \mathcal{O}_K$  be the order of conductor  $c$ .

Let  $E/\mathbb{C}$  be an elliptic curve with  $\text{End } E \cong \mathcal{O}_c$  and choose a Weierstrass equation

$$E : y^2 = x^3 + Ax + B$$

for  $E$  such that  $A, B \in \mathbb{Q}(j(E))$ .

Let  $\Lambda = \langle \omega_1, \omega_2 \rangle$  be the period lattice of  $E$ , ordered so that  $\Im(\omega_2/\omega_1) > 0$ , and let  $\wp_\Lambda(z)$  be the corresponding Weierstrass  $\wp$ -function. Then

$$K[c] = K \left( j\left(\frac{\omega_2}{\omega_1}\right), \wp_\Lambda\left(\frac{1}{c}\Lambda - \Lambda\right) \right).$$

Shimura and Taniyama generalized this theory to CM fields  $K$ .

## Part II

# Unit groups

## 4 Elliptic units

Harder question: Can we analytically construct *units* of  $K$ ?

Yes, in the case where  $K$  is imaginary quadratic.

**Definition 1.** A *modular unit* for a congruence subgroup  $\Gamma$  of  $\mathrm{SL}_2(\mathbb{Z})$  is a meromorphic function on  $X(\Gamma)$  whose divisor is supported on the cusps of  $X(\Gamma)$ .

Mantra: CM values of modular units are units of class fields of imaginary quadratic fields.

Units so constructed are called *elliptic units* or *Siegel units*.

Set

$$\Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24},$$

$$\alpha(z) = \frac{\Delta(2z)^2}{\Delta(4z)\Delta(z)}.$$

The functions  $\Delta(Nz)/\Delta(z)$  and  $\alpha$  are examples of modular units.

Suppose  $\tau \in \mathcal{H}$  is a point such that

$$\mathcal{O}_\tau := \text{End } \mathbb{C}/\langle 1, \tau \rangle$$

is the order in  $K$  of conductor  $c$ . Let  $K(c)$  be the associated *ring class field*. Then:

- $\Delta(N\tau)/\Delta(\tau) \in K(c)$
- $N^{24}\Delta(N\tau)/\Delta(\tau)$  is a unit in  $K(c)$
- $\alpha(\tau)$  is a unit in  $K(c)$

## 5 Generalizations of the elliptic unit construction?

Can one use the CM theory of Shimura and Taniyama to produce units of class fields of CM fields in an analogous way?

It seems not :(

Fundamental obstruction: There are no modular units on Hilbert modular varieties! (The boundary components in the minimal compactification have codimension  $\geq 2$ .)

But...

## 6 Stark's conjecture and Stark units

...there is a well known conjecture due to Stark (1970s) concerning units in general number fields. Stark's conjecture is a statement of the following shape:

$$(\text{derivative of a partial zeta function})|_{s=0} = \log |\text{unit}|.$$

The units for which these formulas are valid are called *Stark units*.

Stark units are special units which are intimately related to  $L$ -values.

The conjecture suggests that it may be easier to get at the logarithm of a unit rather than at the unit itself.

## 7 Logarithms of units and periods of Eisenstein series

Logarithms of elliptic units:

$$d\log \Delta(z) = -24E_2(z)dz =: \omega_{\text{Eis}}$$

Therefore,  $\log(\Delta(N\tau)/\Delta(\tau))$  may be viewed as a period of an Eisenstein series.

Although, in higher dimensions, there are no modular units, there are plenty of Eisenstein series!

Charolloix: In appropriate situations, Stark's conjecture implies that  $\log |\text{product of Stark units}| = \text{period of Eisenstein series}$ .

Charolloix-Darmon: One can get information about the argument of the product of Stark units appearing on the left hand side. This leads to a refinement of Stark's conjecture.

## 8 The Charolloix-Darmon construction

I want to be more specific: Which Eisenstein series? What cycles are we integrating over to get the periods?

I want to develop these ideas enough to give a conjectural formula (due to Charolloix-Darmon) for a global unit in terms of periods of Eisenstein series.

Let  $F$  be a totally real field of degree  $n := [F : \mathbb{Q}] \geq 3$  and narrow class number  $h_F^+ = 1$ . Let  $v_1, \dots, v_n$  be the infinite places of  $F$ . For  $x \in F$ , we write  $x_i$  for  $v_i(x)$ . Let  $\delta$  be a totally positive generator of the different of  $F/\mathbb{Q}$ .

## 8.1 Eisenstein series

Define

$$E_2(z_1, \dots, z_n) = \zeta_F(-1) + 2^n \sum_{\substack{\mu \in \mathcal{O}_F \\ \mu > 0}} \sigma_1(\mu) e^{2\pi i \left( \frac{\mu_1}{\delta_1} z_1 + \dots + \frac{\mu_n}{\delta_n} z_n \right)},$$

where

$$\sigma_1(\mu) = \sum_{(\nu) | (\mu)} |N_{F/\mathbb{Q}}(\nu)|.$$

Set

$$\omega_{\text{Eis}} = \frac{(2\pi i)^n}{\sqrt{d_F}} E_2(z_1, \dots, z_n) dz_1 \cdots dz_n.$$

The form  $\omega_{\text{Eis}}$  is a closed, holomorphic  $n$ -form on the Hilbert modular variety

$$\mathcal{X} := \mathcal{H}^n / \text{SL}_2(\mathcal{O}_F).$$

### 8.1.1 Projection to the +-eigenspace

Let  $W_{v_1}$  be the involution on  $\mathcal{X}$  induced by the map  $\mathcal{H}^n \rightarrow \mathcal{H}^n$  given by

$$(z_1, z_2, \dots, z_n) \mapsto (-\bar{z}_1, z_2, \dots, z_n).$$

Let  $\omega_{\text{Eis}}^+$  be the projection of  $\omega_{\text{Eis}}$  onto the + eigenspace for  $W_{v_1}^*$ :

$$\omega_{\text{Eis}}^+ = \frac{1}{2}(\omega_{\text{Eis}} + W_{v_1}^* \omega_{\text{Eis}}).$$

Fact:  $\omega_{\text{Eis}}^+$  is *exact*.

## 8.2 ATR points and cycles

We now wish to construct a cycle which will play the role of a CM point.

Let  $K$  be an *almost totally real (ATR)* quadratic extension of  $F$ , i.e., there is a unique infinite place of  $F$ , say  $v_1$ , which extends to a complex place  $v'_1$  of  $K$ .

Let  $\tau \in K$  be such that  $v'_1(\tau) \in \mathcal{H}$ .

For  $i = 2, \dots, n$ , let  $v'_i, v''_i$  be the extensions of  $v_i$  to  $K$  and set

$$\Upsilon_i = \text{geodesic in } \mathcal{H} \text{ from } v'_i(\tau) \text{ to } v''_i(\tau),$$

$$R_\tau = \{v'_1(\tau)\} \times \Upsilon_2 \cdots \Upsilon_n \subset \mathcal{H}^n.$$

To the point  $\tau$  we may associate the  $\mathcal{O}_F$ -order

$$\mathcal{O}_\tau := \{\gamma \in M_2(\mathcal{O}_F) \cap GL_2(F) : \gamma\tau = \tau\} \cup \{0\} \hookrightarrow K.$$

The stabilizer  $SL_2(\mathcal{O}_F)_\tau$  can be identified with the group  $U$  defined by the exact sequence

$$1 \longrightarrow U \longrightarrow \mathcal{O}_\tau^\times \xrightarrow{N_{K/F}} \mathcal{O}_F^\times \longrightarrow 1.$$

Therefore, by Dirichlet's unit theorem,

$$\text{rank } SL_2(\mathcal{O}_F)_\tau = n - 1.$$

This group acts properly on  $R_\tau \approx \mathbb{R}^{n-1}$  with compact quotient homeomorphic to  $\mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$ . Thus, we obtain an  $(n-1)$ -cycle

$$\Delta_\tau : SL_2(\mathcal{O}_F) \backslash R_\tau \hookrightarrow SL_2(\mathcal{O}_F) \backslash \mathcal{H}^n = \mathcal{X}.$$

The reason we consider ATR extensions  $K$  is so that this numerology works out!

**Lemma.** The class  $[\Delta_\tau] \in H_{n-1}(\mathcal{X}, \mathbb{Z})$  is torsion.

*Proof.* We exploit known facts about the (co)homology of Hilbert modular varieties.

If  $n - 1$  is odd, then  $H_{n-1}(\mathcal{X}, \mathbb{Z})$  is finite.

If  $n - 1$  is even, then  $H^{n-1}(\mathcal{X}, \mathbb{C})$  is spanned by the cohomology classes of the closed,  $\mathrm{GL}_2(\mathbb{R})^n$ -invariant forms

$$\eta_S := \bigwedge_{i \in S} \frac{dz_i \wedge d\bar{z}_i}{y_i^2}, \quad S \subset \{1, \dots, n\}, \quad |S| = \frac{n-1}{2}.$$

Since the projection of  $R_\tau$  onto each factor  $\mathcal{H}$  has dimension 0 or 1 and the pullback of  $\eta_S$  to each such factor has dimension 0 or 2, it follows that

$$\int_{\Delta_\tau} \eta_S = 0.$$

The lemma follows by duality. □

Let  $e$  be the exponent of  $H_{n-1}(\mathcal{X}, \mathbb{Z})_{tors}$ . Then there is an  $n$ -chain  $C_\tau$  on  $\mathcal{X}$ , well defined up to  $n$ -cycles, such that

$$\partial C_\tau = e\Delta_\tau.$$

As  $\omega_{\text{Eis}}^+$  is exact,

$$\frac{1}{e} \int_{C_\tau} \omega_{\text{Eis}}^+$$

does not depend on our choice of  $C_\tau$  by Stokes' theorem.

In his thesis, Charolloix shows that this integral is the value at 0 of the derivative of a partial  $L$ -function of the type appearing in Stark's conjecture.

Thus, in the situation of ATR extensions, he is able to rephrase Stark's conjecture as follows:

Let  $I \subset \mathcal{O}_F$  be the conductor of the order  $\mathcal{O}_\tau$ . Then there is a unit  $u_\tau$  of the ring class field  $K(I)$  such that

$$\begin{aligned} (-2)^{n-1} \frac{1}{e} \int_{C_\tau} \omega_{\text{Eis}}^+ &= \delta_I (2\pi i)^{n-1} \log |\tilde{v}_1(u_\tau)| \\ &= \delta_I (2\pi i)^{n-1} \Re(\log \tilde{v}_1(u_\tau)). \end{aligned}$$

(Here,  $\delta_I$  is the size of a certain subquotient of  $\mathcal{O}_K^\times$ .)

To get information about  $\arg \tilde{v}_1(u_\tau) = \Im(\log \tilde{v}_1(u_\tau))$ , Charolloix and Darmon replace  $\omega_{\text{Eis}}^+$  “ $= \Re_{v_1} \omega_{\text{Eis}}$ ” with  $\omega_{\text{Eis}}$  itself.

But now we have to be careful:  $\omega_{\text{Eis}}$  is not exact and  $C_\tau$  is only well defined up to  $n$ -cycles, so

$$\frac{1}{e} \int_{C_\tau} \omega_{\text{Eis}}$$

is only well defined up to  $e^{-1}\Lambda_{\text{Eis}}$ , where

$$\Lambda_{\text{Eis}} = \left\{ \int_{\Delta} \omega_{\text{Eis}} : \Delta \text{ an } n\text{-cycle on } \mathcal{X} \right\} \subset \mathbb{C}.$$

**Proposition.** The group  $\Lambda_{\text{Eis}}$  has rank one and is contained in  $(2\pi i)^n \mathbb{Q}$ .

Let  $\Lambda'_{\text{Eis}}$  be the lattice in  $(2\pi i)^n \mathbb{R}$  generated by  $(2\pi i)^n \mathbb{Z}$  and  $e^{-1}\Lambda_{\text{Eis}}$ . Then  $[\Lambda'_{\text{Eis}} : (2\pi i)^n \mathbb{Z}]$  is finite.

**Conjecture.** (Charolloix-Darmon) There is a unit  $u_\tau$  of the ring class field  $K(I)$  such that

$$(-2)^{n-1} \frac{1}{e} \int_{C_\tau} \omega_{\text{Eis}} = \delta_I (2\pi i)^{n-1} \log \tilde{v}_1(u_\tau) \pmod{\Lambda'_{\text{Eis}}}.$$

## Part III

# Mordell-Weil groups

## 9 Heegner points

Let  $E/\mathbb{Q}$  be an elliptic curve of conductor  $N$ .

By the modularity theorem of Wiles, there is a nonconstant, dominant morphism

$$\Phi_E : X_0(N) \longrightarrow E$$

which is defined over  $\mathbb{Q}$ .

Let  $\omega_E$  an invariant differential on  $E/\mathbb{Q}$ .

Base change to  $\mathbb{C}$ , let  $\pi : \mathcal{H}^* \rightarrow X_0(N)(\mathbb{C})$  be the standard projection, and define  $f(z)$  by

$$2\pi i f(z) dz = \pi^* \Phi^* \omega_E.$$

Then  $f(z)$  is a newform in  $S_2(\Gamma_0(N))$  and has rational Fourier coefficients.

For an arbitrary base point  $\tau \in \mathcal{H}^*$ , let

$$\Lambda = \left\{ \int_{\tau}^{\gamma\tau} 2\pi i f(z) dz : \gamma \in \Gamma_0(N) \right\}.$$

Then  $\Lambda$  is a lattice in  $\mathbb{C}$  and the map

$$\mathcal{H}^* \rightarrow \mathbb{C}/\Lambda \rightarrow E(\mathbb{C}), \quad \tau \mapsto \wp_{\Lambda} \left( \int_{\infty}^{\tau} 2\pi i f(z) dz \right)$$

is just the composition  $\Phi_E \circ \pi$ .

Let  $K$  be an imaginary quadratic field and let  $\tau \in \mathcal{H} \cap K$ .

The elliptic curve  $\mathbb{C}/\langle 1, \tau \rangle$  has CM by an order in  $K$ . Let  $c$  be its conductor.

By CM theory,

$$\pi(\tau) \in X_0(N)(K(c)),$$

where  $K(c)$  is the ring class field of  $K$  of conductor  $c$ .

As  $\Phi_E$  is defined over  $\mathbb{Q}$ ,

$$u_\tau := \Phi_E(\pi(\tau)) \in E(K(c)).$$

The point  $u_\tau$  is called a *Heegner point*. These Heegner points are the elliptic curve analogues of the elliptic units discussed earlier.

Let

$$\log_E : E(\mathbb{C}) \longrightarrow \mathbb{C}/\Lambda_f = \mathbb{C}/(\text{periods of } E(\mathbb{C})).$$

be the inverse of  $\wp_\Lambda$ . Then we can write

$$\log_E u_\tau = \int_\infty^\tau 2\pi i f(z) dz,$$

Read: Logarithms of Heegner points are integrals of cusp forms.

We had a formula of the same form with the Heegner point replaced by an elliptic unit and the cusp form replaced by an Eisenstein series!

How far can we push this analogy? Are there analogues of Stark units for Mordell-Weil groups? What about analogues of the Darmon-Charolloix conjecture?

## 10 ATR points

Let  $F$  be a totally real field of degree  $n := [F : \mathbb{Q}] \geq 2$  and narrow class number  $h_F^+ = 1$ . Let  $v_1, \dots, v_n$  be the infinite places of  $F$ .

Let  $E/F$  be an elliptic curve of everywhere good reduction<sup>a</sup>, and assume that  $E/F$  is *modular*, i.e., there is a Hilbert modular form  $f \in S_2(\mathrm{SL}_2(\mathcal{O}_F))$  such that  $L(E/F, s) = L(f, s)$ .

Let

$$\omega_f = (2\pi i)^n f(z_1, \dots, z_n) dz_1 \cdots dz_n.$$

Then  $\omega_f$  is a holomorphic  $n$ -form on the Hilbert modular variety

$$\mathcal{X} := \mathrm{SL}_2(\mathcal{O}_F) \backslash \mathcal{H}^n.$$

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<sup>a</sup>for simplicity, and because it's particularly interesting

## 10.1 Projection to eigenspaces for complex conjugation

There are involutions  $W_{v_1}, \dots, W_{v_n}$  of  $\mathcal{X}$  corresponding to the maps  $z_i \mapsto -\bar{z}_i$ .

Let  $\omega_f^+$  be the projection of  $\omega_f$  onto the +1-eigenspace for each involution  $W_{v_i}$ .

**Remark.** The form  $\omega_f^+$  plays the role of  $\omega_{\text{Eis}}$ , not  $\omega_{\text{Eis}}^+$ . We could perform this construction with any projection of  $\omega_f$  onto a simultaneous eigenspace of the  $W_{v_i}$ ; we choose the +1-eigenspaces for convenience. These  $2^n$  extra degrees of freedom in the construction come from the fact that cuspidal systems of Hecke eigenvalues occur in  $H^n(\mathcal{X}, \mathbb{C})$  with multiplicity  $2^n$  while Eisenstein series occur with multiplicity one.

## 10.2 Cycles and periods

Let  $K/F$  be an ATR extension such that the extension  $\tilde{v}_1$  of  $v_1$  to  $K$  is complex.

Let  $\tau \in K$  be such that  $\tilde{v}_1(\tau) \in \mathcal{H}$  and let  $\Delta_\tau$  be the associated  $(n-1)$ -cycle, as before. Then

$$[\Delta_\tau] \in H_{n-1}(\mathcal{X}, \mathbb{Z})$$

is torsion, so there is an  $n$ -chain  $C_\tau$ , well defined up to  $n$ -cycles, such that

$$\partial C_\tau = e\Delta_\tau,$$

where  $e$  is the exponent of  $H_{n-1}(\mathcal{X}, \mathbb{Z})_{\text{tors}}$ .

Therefore, the quantity

$$\frac{1}{e} \int_{C_\tau} \omega_f^+$$

is well defined up to elements of

$$\Lambda_f^+ := \left\{ \int_{\Delta} \omega_f^+ : \Delta \text{ an } n\text{-cycle on } \mathcal{X} \right\}.$$

Let  $\Lambda_1$  be the period lattice of  $E \times_{v_1} \text{Spec } \mathbb{C}$  and let  $\Omega_i^+$ ,  $i = 2, \dots, n$ , be the real periods associated to  $E \times_{v_i} \mathbb{C}$  and an invariant differential  $\omega_E$  on  $E/F$ . Let

$$\Lambda_E^+ = \Omega_2^+ \cdots \Omega_n^+ \Lambda_1.$$

**Conjecture.** (Oda).  $\Lambda_f^+$  is a lattice commensurable with  $\Lambda_E^+$ .

Let

$$\log_E^+ : (E \times_{v_1} \text{Spec } \mathbb{C})(\mathbb{C}) \longrightarrow \mathbb{C}/\Lambda_1 \longrightarrow \mathbb{C}/\Lambda_E^+$$

be the inverse of the Weierstrass parametrization composed with multiplication by  $\Omega_2^+ \cdots \Omega_n^+$ .

Let  $I$  be the conductor of the order associated to  $\tau$ .

**Conjecture.** (Darmon) There is a point  $u_\tau^+ \in E(K(I))$  such that

$$\log_E^+ \tilde{v}_1(u_\tau^+) = \frac{1}{e} \int_{C_\tau} \omega_f^+ \pmod{\Lambda_f^+ + \Lambda_E^+}.$$

## 10.3 Compatibility with BSD

So we have a conjecture asserting the existence of a systematic supply of points on  $E$  defined over ring class fields of the ATR field  $K$ .

What does BSD have to say about the rank growth of  $E$  in this tower of number fields?

Good reduction is stable, so  $E$  has everywhere good reduction over  $K$ . It follows that the sign in the functional equation of  $L(E/K, s)$  is

$$(-1)^{\text{number of infinite places of } K} = (-1)^{2(n-1)+1} = -1.$$

The  $L$ -function  $L(E/K(I), s)$  factors as

$$L(E/K(I), s) = \prod_{\chi: \text{Gal } K(I)/K \rightarrow \mathbb{C}^\times} L(E/K, \chi, s)$$

Each twisted  $L$ -function  $L(E/K, \chi, s)$  satisfies a functional equation in its own right which is symmetric and has the same sign as the functional equation of  $L(E/K, s)$  —  $-1$ .

Therefore,

$$\text{rank } E(K(I)) \stackrel{\text{BSD}}{=} \text{ord}_{s=1} L(E/K(I), s) \geq [K(I) : K].$$

So the existence of a systematic supply of points on  $E$  algebraic over ring class fields of  $K$  is compatible with the rank heuristic provided by BSD and signs in functional equations.

# 11 Beyond ATR fields

In deriving the above rank growth heuristic, we only used the assumption that  $K$  is ATR to guarantee that the sign in the functional equation of  $E/K$  is  $-1$ .

But there are many more non-ATR fields  $K$  with this property.

Are there (conjectural) point constructions explaining the rank growth in these cases?

(G, 2007) Yes! Idea: Replace Hilbert modular varieties with quaternionic Shimura varieties. This is analogous to enriching our supply of modular parametrizations of elliptic curves over  $\mathbb{Q}$  by admitting parametrizations by Shimura curves in addition to those by classical modular curves.

These constructions are  $p$ -adic and generalize a construction of Darmon for real quadratic fields  $K$  satisfying a Heegner hypothesis.