

Geometric Weil Rep and Applications

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Preliminaries:

$k = \mathbb{F}_q$ (q odd). Fix an additive character $\psi : k \rightarrow \mathbb{C}^\times$.

Fourier Transform. $\mathcal{F} : L^2(k, \mathbb{C}) \rightarrow L^2(k^*, \mathbb{C})$.

$$\mathcal{F}(f)(\xi) = \nu \cdot \sum_{l \in k} f(l) \psi(\langle \xi, l \rangle)$$

Usual normalization: $\nu = \frac{1}{\sqrt{q}}$.

Letter : Deligne \rightarrow Kazhdan: $\nu = \frac{1}{\sum_{z \in k} \psi(-\frac{1}{2}z^2)}$.

Question: Why?

(V, ω) = symplectic vector space over k , $\dim V = 2N$.

$H = H(V, \omega) = V \times k$: Heisenberg group; $Z := Z(H) = 0 \times k$.

$$(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2}\omega(v, v')).$$

Theorem (Stone-von Neumann): There exists a *unique* irreducible finite-dimensional representation (π, \mathcal{H}) of H with central character ψ .

The Weyl Transform:

Let $\mathcal{S}(H, \psi) = \{f : H \rightarrow \mathbb{C} \mid \forall z \in Z, h \in H, f(z \cdot h) = \psi(z)^{-1}f(h)\}$.

Theorem. The map $\widehat{\cdot} : \text{End}(\mathcal{H}) \rightarrow \mathcal{S}(H, \psi)$ given by

$$A \rightarrow \widehat{A}(h) = \frac{1}{\dim \mathcal{H}} \text{Tr}(A\pi(h^{-1}))$$

is an isomorphism of algebras (i.e., $\widehat{A \cdot B} = \widehat{A} * \widehat{B}$), with inverse

$$f \rightarrow \frac{1}{|Z(H)|} \sum_{h \in H} f(h)\pi(h)$$

Fact I: $\mathcal{S}(H, \psi) \simeq \mathcal{S}(V)$ where

$$f * g(v) = \sum_{v_1+v_2=v} \psi(-\frac{1}{2}\omega(v_1, v_2))f(v_1)g(v_2).$$

Let $G = \text{Sp}(V)$. For any $g \in G$, set $\pi^g(h) := \pi(g \cdot h)$.

Fact II: $\pi^g \simeq \pi$, hence $\pi(g \cdot h) = \rho(g)\pi(h)\rho(g^{-1})$, $\rho(g)$ unique up to a scalar.

Fact III: The projective representation ρ can in fact be linearized.

The Kernel $K_g(v)$:

For $g \in \mathrm{Sp}(V)$, $v \in V$, set $K_g = \widehat{\rho(g)} \in \mathcal{S}(V)$.

We have $(m \times \mathrm{Id})^* K = (p_1 \times \mathrm{Id})^* K * (p_2 \times \mathrm{Id})^* K$.

Set $U = \{g \in \mathrm{Sp}(V) : g - \mathrm{I} \text{ is invertible}\}$.

Let $\kappa : U \rightarrow \mathfrak{sp}(V)$ be the Cayley transform $\kappa(g) = \frac{g+\mathrm{I}}{g-\mathrm{I}}$.

$e := \sum_{z \in k} \psi(z^2)$ $\sigma : k^\times \rightarrow \mathbb{C}^\times$ Legendre character

Define (for $g \in U$):

$$K_g(v) = \frac{e^{2N}}{q^{2N}} \sigma(\det(\mathrm{I} + \kappa(g))) \psi\left(\frac{1}{4} \omega(\kappa(g)v, v)\right)$$

Theorem. There exists a unique multiplicative extension of $K_g(v)$ to $\mathrm{Sp}(V)$.

Remark. Uniqueness follows immediately from $U \cdot U = G$.

Corollary. The character of the Weil representation is given by

$$\mathrm{ch}_\rho(g) = \frac{e^{2N}}{q^N} \sigma(\det(\mathrm{I} + \kappa(g)))$$

Sheaf-Function Dictionary

Fix $\overline{\mathbb{Q}}_l \simeq \mathbb{C}$.

\mathbf{X} = variety/ k

$Fr : \mathbf{X} \rightarrow \mathbf{X}$ (Frobenius)

$X = \mathbf{X}(k) = \mathbf{X}^{Fr}$

$D^b(\mathbf{X})$ = bounded derived category of constructible l -adic sheaves.

Take $\mathcal{F} \in D^b(\mathbf{X})$.

$H^i(\mathcal{F}_x)$ = stalk cohomology.

Weil structure $Fr : Fr^* \mathcal{F} \rightarrow \mathcal{F}$.

Define $x \in X \implies f^{\mathcal{F}}(x) = \sum_i (-1)^i \text{Tr}(Fr|_{H^i(\mathcal{F}_x)})$.

Obvious approach:

$K_u(v) \Rightarrow \mathcal{K}_{\mathbf{U}} \in D^b(\mathbf{U}) \Rightarrow \mathcal{K}_{\mathbf{G}} \in D^b(\mathbf{G}) \Rightarrow K_g(v)$.

$\mathcal{K}_{\mathbf{U}} = \mathcal{C} \otimes \mathcal{K}'_{\mathbf{U}}$ where:

a) $\mathcal{K}'_{\mathbf{U}} = i^{-1} \mathcal{L}_{\psi(\frac{1}{4}\omega(av, v))}$ where $i : \mathbf{U} \times \mathbf{V} \rightarrow \text{End}(\mathbf{V}) \times \mathbf{V}$.

b) $\mathcal{C} = \mathcal{E}^{\otimes 2 \otimes 2N} \otimes \mathcal{L}_{\sigma(\det(I + \kappa(g)))}[4N](2N)$

c) \mathcal{E} is defined by $f^{\mathcal{E}} = e$.

Now

$$\mathcal{K}_{\mathbf{G}} = j_{!*} \mathcal{K}_{\mathbf{U}}$$

where $j : \mathbf{U} \rightarrow \mathbf{G}$ and

$$j_{!*} \mathcal{K} = \text{im}[P_j \mathcal{K} \mapsto P_{j*} \mathcal{K}]$$

Application: Quantum unique ergodicity

Physical backdrop Eigenstates of a quantum system are equidistributed in the semiclassical limit $\hbar \mapsto 0$.

Classical Torus: (\mathbb{T}, ω) two dimensional symplectic torus, i.e. $\mathbb{T} = W/\Lambda$ where $W \simeq \mathbb{R}^2$ and Λ is a rank two lattice.

We assume ω is integral, i.e. $\omega : \Lambda \times \Lambda \rightarrow \mathbb{Z}$ and $\text{Vol}(\mathbb{T}) = 1$.

$$\Lambda^* = \{\xi \in W^* \mid \xi(\Lambda) \subseteq \mathbb{Z}\}.$$

$$\Gamma = \{A \in \text{SL}_2(\mathbb{R}) \mid \Gamma(\Lambda) \subseteq \Lambda\}$$

Birkhoff Ergodic Theorem: For a hyperbolic element $A \in \Gamma$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{r=1}^N f(A^r x) = \int_{\mathbb{T}} f |\omega|$$

Weyl Quantization:

The Rieffel Torus: $\mathcal{A}_{\hbar} = \mathbb{C} \langle s(\xi) : \xi \in \Lambda^* \rangle / \sim$ where
 $\sim = s(\xi + \eta) = e^{\pi i \hbar \omega(\xi, \eta)} s(\xi) s(\eta)$

$$\hbar = 0 \implies \mathcal{A}_{\hbar} = \mathbb{C}[\Lambda^*]$$

In fact the map $s : \Lambda^* \rightarrow \mathcal{A}_{\hbar}$ gives rise to a standard basis of \mathcal{A}_{\hbar} .

Obviously, Γ acts on \mathcal{A}_{\hbar} : $f \mapsto f^B$.

Equivariant quantization of the torus:

$$\pi_{\hbar} : \mathcal{A}_{\hbar} \rightarrow \text{End}(\mathcal{H}_{\hbar})$$

$$\rho : \Gamma \rightarrow \text{PGL}(\mathcal{H}_{\hbar})$$

where

$$\rho_{\hbar}(B) \pi_{\hbar}(f) \rho_{\hbar}(B)^{-1} = \pi_{\hbar}(f^B)$$

Theorem: For $\hbar = \frac{1}{p}$, there exists a unique irrep of \mathcal{A}_{\hbar} fixed under the action of Γ (on $\text{Irr}(\mathcal{A}_{\hbar})$).

The intertwiners give the projective representation ρ_{\hbar} of Γ .

Kurlberg-Rudnick Conjecture:

Let Γ_ρ be $\Gamma \bmod \rho$. Then $\Gamma_\rho \simeq \mathrm{SL}_2(\mathbb{F}_\rho)$.

Let $A \in \Gamma$ be hyperbolic and $T_A = \mathrm{Cent}_{\mathrm{SL}_2(\mathbb{F}_\rho)}(A)$ (Hecke torus)

T_A acts semisimply on \mathcal{H}_{\hbar} , hence we have eigenspaces Ψ .

Theorem. If $\hbar = \frac{1}{\rho}$, for ρ sufficiently large prime, then for every $f \in \mathcal{A}_{\hbar}$ and Ψ ,

$$\left| \frac{1}{|T_A|} \sum_{B \in T_A} \langle \Psi | \pi_{\hbar}(f^B) \Psi \rangle - \int_{\mathbb{T}} f | \omega | \right| \leq \frac{C_f}{\sqrt{\rho}}$$