Artin L-functions

1. Dirichlet L-series Let K be an algebraic number field.

$$\begin{aligned} \mathfrak{m} = &\text{ ideal of } K \text{ (which means, an ideal of } O_k) \\ I(\mathfrak{m}) = &\text{ the group of fractional ideals prime to } \mathfrak{m} \\ \chi = &\text{ character of } I(\mathfrak{m}) \end{aligned}$$

We assume that χ is trivial on some subgroup of $I(\mathfrak{m})$ of finite index, (e.g such as the group of principal ideals prime to $\mathfrak{m}$.)

Define $\chi(\mathfrak{a}) = 0$ if $\mathfrak{a}$ is an ideal of K not prime to $\mathfrak{m}$. Define $N\mathfrak{a} = N_{K/\mathbb{Q}}\mathfrak{a}$.

The Dirichlet L-series is defined by

$$L(s, \chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{(N\mathfrak{a})^s} = \prod_{\mathfrak{p}} \frac{1}{1 - \frac{\chi(\mathfrak{p})}{(N\mathfrak{p})^s}}$$

If χ is the trivial character, then $L(s,\chi) = \zeta_K(s)$, the Dedekind zeta-function.$L(s,\chi)$ converges for $Re(s) > 1$ and can be analytically continued to $\mathbb{C}$, and is an entire function if χ is not trivial.

$$\log L(s,\chi) = \sum_{\mathfrak{p},n} \frac{\chi(\mathfrak{p}^n)}{(N\mathfrak{p})^{ns} n}$$

E/K Galois extension of number fields, Galois group $G(E/K) = G$.

From class-field theory, if $G(E/K)$ is abelian, then for suitable $\mathfrak{m}$, $I(\mathfrak{m})/P(\mathfrak{m})$ is isomorphic to $G(E/K)$, via the Frobenius map. Then χ can be identified with a character of $G(E/K)$. 2. Artin L-series Let $\rho : G \to M_m(\mathbb{C})$ be a representation, with character χ. For a prime ideal $\mathfrak{p}$ of K, let $\mathfrak{P}$ be a prime of E lying over $\mathfrak{p}$.

$$D_{\mathfrak{P}} = \{\sigma \in G : \sigma\mathfrak{P} = \mathfrak{P}\}, \text{ the decomposition group}$$

$$I_{\mathfrak{P}} = \{\sigma \in D_{\mathfrak{P}} : \sigma a \equiv a \text{ mod } \mathfrak{P}, a \in O_E\}, \text{ the intertia group}$$

$$\sigma_{\mathfrak{P}} \in D_{\mathfrak{P}}, \text{ the Frobenius}, \quad \sigma_{\mathfrak{P}}(a) \equiv a^{N\mathfrak{p}} \text{ mod } \mathfrak{P}, a \in O_E$$

Artin L-series $\log L(s, \chi, E/K)$ can be defined by

$$\log L(s, \chi, E/K) = \sum_{n,\mathfrak{p}} \frac{\chi(\mathfrak{p}^n)}{(N\mathfrak{p})^{ns} n}$$

$$\text{where} \quad \chi(\mathfrak{p}^n) = \frac{1}{e} \sum_{\tau \in I_{\mathfrak{P}}} \chi(\sigma_{\mathfrak{P}}^n \tau)$$

where $e = e(E/K)$ is the ramification index. When $\mathfrak{p}$ is unramified, then $\chi(\mathfrak{p}^n) = \chi(\sigma_{\mathfrak{P}}^n)$. $\log L(s, \chi, E/K)$ converges for $Re(s) > 1$.

The definition depends on a choice of $\mathfrak{P}$ lying over $\mathfrak{p}$; a different choice gives a conjugate Frobenius $\sigma_{\mathfrak{P}}$.

Note that if $G(E/K)$ is abelian, then $\log L(s, \chi, E/K)$ is the same as the Dirichlet L-series defined previously. Note that if χ is the trivial character χ_0, then $L(s, \chi_0, E/K) = \zeta_K(s)$.

Another point of view: define

$$E = E_{\mathfrak{P}} = \frac{1}{e} \sum_{\tau \in I_{\mathfrak{P}}} \tau \in \mathbb{C}G.$$

This is an idempotent element in the group algebra $\mathbb{C}G$. Extend the representation ρ of G to $\mathbb{C}G$ linearly. Define

$$\rho(\mathfrak{p}^n) = \rho(\sigma_{\mathfrak{P}}^n E_{\mathfrak{P}}) \in M_m(\mathbb{C}).$$

Define

$$\log L(s, \rho, E/K) = \sum_{n,\mathfrak{p}} \frac{1}{(N\mathfrak{p})^{ns} n} \rho(\mathfrak{p}^n)$$

so $\log L$ is a matrix-valued function, whose trace is the previous version. Exponentiate; use

$$\text{trace}\left(\sum_{n=1}^{\infty} A^n t^n / n\right) = -\log(\det(1 - At))$$

where $A = \rho(\mathfrak{p})$, $t = 1/(N\mathfrak{p})^s$. Get

$$L(s, \chi, N/K) = \prod_{\mathfrak{p}} \frac{1}{\det\left(I - \frac{1}{(N\mathfrak{p})^s}\rho(\mathfrak{p})\right)}$$

Properties.

I. $L(s, \chi_1 + \chi_2, E/K) = L(s, \chi_1, E/K)L(s, \chi_2, E/K)$

This is because $\log L(s, \chi, E/K)$ is linear in χ.

II. Suppose that $E \subset F$, and F/K is Galois with group $\overline{G}$. As above, χ is a character of G. View χ as a characer of $\overline{G}$. Then

$$L(s, \chi, E/K) = L(s, \chi, F/K)$$

To prove this requires a small amount of book-keeping, because there will be a new Frobenius and new decomposition and inertia groups for F/K.

Recall the theory of induced characters.

Suppose that H is a subgroup of G, and that we have a represntaion $H \to M_k(\mathbb{C})$ with character χ. Then there is a representation of G with character $\mathrm{ind}_H^G \chi$, whose given by

$$\mathrm{ind}_H^G \chi(g) = (1/|H|) \sum_{x \in G} \dot{\chi}(x^{-1}gx)$$

where $\dot{\chi}(h) = \chi(h)$ if $h \in H$, 0 otherwise.

III Suppose that $K \subset F \subset E$, χ is a character of $G(E/F) = H$. Then $\operatorname{ind}_H^G \chi$ is a character of $G = G(E/K)$, and

$$L(s, \chi, E/F) = L(s, \operatorname{ind}_H^G \chi, E/K)$$

Artin's Induction Theorem: If χ is a character of G, then

$$\chi = \frac{1}{|G|} \sum_i a_i \operatorname{ind}_{H_i}^G \chi_i$$

where χ_i is a character of H_i and H_i is cyclic, $a_i \in \mathbb{Z}$.

We are using $G = G(E/K)$. Let E_i be the fixed field of H_i. Then

$$
\begin{aligned}
L(s,\chi,E/K)^{|G|} &= \prod_i L(s,\operatorname{ind}_{H_i}^{G}\chi_i,E/K)^{a_i} \\
&= \prod_i L(s,\chi_i,E/E_i)^{a_i}
\end{aligned}
$$

Each $L(s,\chi_i,E/E_i)$ is a Dirichlet series. So $L(s,\chi,E/K)^{|G|}$ has an analytic continuation to a meromorphic function on $\mathbb{C}$. Artin conjectured that $L(s,\chi,E/K)$ has an analytic continuation to a meromorphic function on $\mathbb{C}$, and indeed to an entire function if χ does not contain the trivial character.

Recall Brauer's Induction Theorem:

$$\chi = \sum_i a_i \operatorname{ind}_{H_i}^G \chi_i$$

where χ_i is a character of degree 1 of the abelian group H_i, $a_i \in \mathbb{Z}$. Again let E_i be the fixed field of H_i. Then

$$L(s, \chi, E/K) = \prod_i L(s, \chi_i, E/E_i)^{a_i}$$

It follows that $L(s, \chi, E/E_i)$ indeed has an analytic continuation to a meromorphic function on $\mathbb{C}$. It has an analytic continuation to an entire function provided that each $a_i > 0$.

This is indeed the case if G is such that each irreducible character of G has the form $\mathrm{ind}_H^G \chi$ where χ has degree 1. This is the case, for example, if G is a finite p-group, or for various small groups such as S_3, S_4, or dihedral groups. The smallest group order for which not all $a_i > 0$ is $G = SL(2,3)$; in this case there is a character χ of degree 2 which is not of the form $\mathrm{ind}_H^G \chi$ where χ has degree 1. For this χ, it has been proved by Tunnell and Langlands that $L(s, \chi, E/K)$ is an entire function. In the case that some $a_i < 0$, one has to consider the zeros of $L(s, \chi_i, E/E_i)$. To my knowledge, this has never been successfully done.

If χ is a non-trivial irreducible character of G, then χ is a $\mathbb{Q}$-linear combination of non-trivial irreducible characters of cyclic subgroups, induced up to G. It then follows that

$$L(1, \chi, E/K) \neq 0$$

One can use the Artin L-function to give a fairly short proof of the Tchebotarev Density Theorem:

$$\sigma \in G; \quad C = \text{ the conjugacy class in } G \text{ containing } \sigma.$$

For a prime ideal $\mathfrak{p}$ of K, let

$$C_{\mathfrak{p}} = \text{ the conjugacy class of all Frobenius } \sigma_{\mathfrak{P}},\ \mathfrak{P} \text{ above } \mathfrak{p}.$$

Then the Dirichlet density of primes $\mathfrak{p}$ of K such that $C_{\mathfrak{p}} = C$ is $|C|/|G|$; that is, $\log(1/s-1)$ is asymptotic to

$$|G|/|C| \sum_{\mathfrak{p}, C_{\mathfrak{p}}=C} \frac{1}{(N\mathfrak{p})^s}$$