The fundamental lemma

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1. The naked lemma

Suppose G to be an unramified reductive group defined over a $\mathfrak{p}$-adic field $\mathfrak{k}$, H an unramified endoscopic group of G. The two have the same rank.

Given a strongly regular semi-simple element t of H, let T_t be its centalizer in H, a maximal torus. There exists a special embedding of T_t in G. The element t gives rise to an element t_G of G, which we assume to be strongly regular in G.

Define orbital integrals of a locally constant function of compact support on any reductive group—for any strongly regular t let

$$\Lambda(f,t) = \Big(\prod_{\Sigma_G} |\alpha(t) - 1|^{1/2}\Big) \left|\frac{\operatorname{vol} T(\mathfrak{o})}{\operatorname{vol} G(\mathfrak{o})}\right| \int_{C_G(t)\backslash G} f(g^{-1}tg)\,\frac{dg}{dt}\,.$$

$$\Lambda(f,t) = \Big(\prod_{\Sigma_G} |\alpha(t)-1|^{1/2}\Big) \left|\frac{\text{vol } T(\mathfrak{o})}{\text{vol } G(\mathfrak{o})}\right| \int_{C_G(t)\backslash G} f(g^{-1}tg)\,\frac{dg}{dt}\,.$$

The ratio of volumes is to eliminate the effect of choice of measures on G and T.

Orbital integrals have something to with fixed-points, since $C_G(t)$ is the set of points on G fixed under conjugation by t. We shall see that this connection extends to something very deep. The product is over roots α with respect to T. Such factors are familiar in fixed point formulas.

The data determining H have something to do a set of elements s in the torus $\widehat{T}$ in the L-group ${}^LG$. They all give rise to the same map κ from the stable conjugacy class of t_G, constant on the ordinary conjugacy classes. The κ-orbital integrals are linear combinations of the ordnary orbital integrals:

$$\Lambda^{\kappa}_{G,H}(f,t) = \sum_{t' \sim t} \kappa(t')\Lambda_G(f,t')\,.$$

If $\kappa \equiv 1$ this is the stable sum $\Lambda^{\mathrm{st}}_G(f,t)$.

Stable conjugacy means conjugacy in the algebrac closure.

Any element f of the Hecke algebra $\mathcal{H}\big(G//G(\mathfrak{o})\big)$ gives rise to an f^H in $\mathcal{H}(H//H(\mathfrak{o})\big)$.

Fundamental Lemma:

$$\Lambda^{\kappa}_{G/H}(f, t_G) = \Lambda^{\mathrm{st}}_{H}(f^H, t)$$

for all strongly G-regular semi-simple elements t of H.

Very roughly, this says that analysis on G invariant under conjugation can be reduced to stable analysis on G and all its endoscopic groups.

Implicit in this are assertions about character sums in G matching stable character sums in H.

2. History

Bill Casselman, 'Notes on $\mathfrak{p}$-adic spherical functions', written as a preface to a new printing of Ian Macdonald's book on spherical functions.

Mark Goresky, Robert Kottwitz, and Robert MacPherson, 'Homology of affine Springer fibres in the unramified case', available on the `http://www.arxiv.org`**.**

Tom Hales, 'The Fundamental Lemma for $\mathrm{Sp}(4)$**',** SL**Proceedings of the American Mathematical Society 125 (1997), 301-308**

`http://www.arxiv.org/math/0312227v2/`

Tom Hales, 'A statement of the Fundamental Lemma', available on the `http://www.arxiv.org`**.**

`http://www.arxiv.org/math/0312227v2/`

David Kazhdan and George Lusztig, 'Fixed point varieties on affine flag manifolds', *Israel Journal of Mathematics* **62 (1988), 129–168; appendix by Joseph Bernstein**

Robert Edward Kottwitz, 'Calculation of some orbital integrals', pages 349–362 in The zeta functions of Picard modular surfaces, CRM.

Jean-Pierre Labesse and Robert Langlands, 'L-indistinguishability for $SL(2)$', *Canadian Journal of Mathematics* **31 (1979), pages 726–785**

Robert P. Langlands, Les débuts d'une formule des traces stables, *Publications mathématiques de l'Université Paris VII*, **1983**

Gerard Laumon, 'On the Fundamental Lemma for unitary groups', audio file at
http://www.fields.utoronto.ca/audio/02-03/shimura/laumon/

Ngo Bao Chau, 'Hitchin fibration and the Fundamental Lemma', talk at the IAS (October 9, 2006):
http://www.math.ias.edu/lg40/ngo.pdf

Jonathan Rogawski, Automorphic representations of unitary groups in three variables, Princeton University Press, 1990.

Jonathan Rogawski and Don Blasius, 'Fundamental Lemmas for $U(3)$ and related groups', Pages 363–394 in The zeta functions of Picard modular surfaces, CRM.

The first result of this type was in the paper by Labesse and Langlands on $\mathrm{SL}(2)$. It was first formulated in complete generality in Langlands' Paris VII booklet. Both of these were motivated by the problems of stable conjugacy arising naturally in the trace formula.

It was proven for $\mathrm{SL}(3)$ by Kottwitz. He calculated orbital integrals by means of geometry on the Bruhat-Tits building of G, following Langlands' work on base change.

The group $\mathrm{SL}(n)$ was dealt with by Waldspurger. Rogawski and others proved it for $\mathrm{U}(3)$, Hales for $\mathrm{Sp}(4)$. I do not know what techniques they used.

A paper by Kazhdan & Lusztig, with an appendix by Bernstein, changed the situation drastically. In this it was pointed out that orbital integrals count points on algebraic varieties over finite fields. Goresky, Kottwitz, and MacPherson used this idea and formulated the Fundamental Lemma in terms of equivariant cohomology. They proved it for unramified conjugacy classes.

Laumon, working at first by himself and then with Ngo, proved it for unitary groups $\mathrm{U}(n)$. It appears that recently Ngo, using somewhat related techniques, has proved it in all cases.

All along, various important technical results and reductions have been added by various people, including Labesse, Langlands, Shelstad, Kottwitz, Waldspurger, Hales, Cunningham, Kazhdan, Lusztig, and Bernstein.

3. The program for this seminar

There are many relatively elementary items to be explained. These include

- **the structure of unramified groups**
- **the Hecke algebras**
- **the L-group (which contains the torus $\widehat{T}$)**
- **the map from the Hecke algebra of G to that of H**
- **characterization and properties of (unramified) endoscopic groups**
- **stable conjugacy and Galois cohomology**
- **the characters κ**
- **orbital integrals and the geometry of buildings**
- **the work of Kazhdan, Lusztig & Bernstein relating orbital integrals to the geometry of the affine Grassmannian**

There are many nastier items to be explained. These include

• why ${}^L H$ does not always embed into ${}^L G$

• specification of the embedding of T_H into G (transfer factors)

• reduction of the general case to that in which f and f^H are Hecke units

• relationship very generally between orbital integrals and varieties over finite fields

• role of equivariant cohomology

• Ngo's use of what he calls the Hitchin fibration

• the transfer from the geometric fields to p-adic groups

• the geometric version of the Satake transform in terms of sheaves on the affine Grassmannian

It would be nice to exhibit some applications, but that's too much to expect.

4. Unramified groups

Split reductive groups over $\mathfrak{k}$ are classified up to isomorphism by root data—quadruples

$$(L, \Sigma, L^\vee, \Sigma^\vee)$$

where L is a lattice, Σ a root system in L, $\Sigma^\vee$ to coroots in the dual lattice $L^\vee$. Here L is to be identified with the group of characters of a maximal split torus T in G. Given a Borel subgroup B we get a based root datum

$$(L, \Delta, L^\vee, \Delta^\vee)\,.$$

The lattice L is the character group $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$, $L^\vee$ the cocharacter group $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$. We may identify $T(\mathfrak{k})$ with $\mathrm{Hom}(L, k^\times)$ or $L^\vee \otimes \mathbb{C}^\times$.

An unramified group has two characterizations. (1) It is obtained by base extension from a smooth reductive group scheme over $\mathfrak{o}$. (2) It is one that splits over an unramfied extension of $\mathfrak{k}$. Essentially because reductive groups over finite fields possess Borel subgroups, so do $G(\mathfrak{o})$ and G. Let it be B, T a torus T contained in B, and A a maximal split torus in T. Such groups are classified through Galois descent theory by quintuples

$$(L, \Sigma, L^\vee, \Sigma^\vee, \sigma)$$

where σ is an automorphism of the based root datum, giving rise to the Galois action on roots.

The lattice L and σ determine the unramified torus T. The embedding of A in T induces an isomorphism of $\mathcal{A} = A/A(\mathfrak{o})$ with $\mathcal{T} = T/T(\mathfrak{o})$. These may both be identified with $X_*(A) = (L^\vee)^\sigma$ via images of the generator ϖ of $\mathfrak{p}$.

The relative Weyl group is the subgroup of the split Weyl group that takes $X_*(A)$ to itself. Elements of W may also be characterized as elements of the split Weyl group fixed by the Frobenius.

5. Satake transform and Hecke algebra

The Hecke algebra is the convolution ring $\mathcal{H}_{\mathbb{Z}}\big(G//G(\mathfrak{o})\big)$. of functions on G of compact support, bi-invariant under $G(\mathfrak{o})$. If (π, V) is an admissible representation of G, it is called un-ramfied if it subspace of vectors fixed by $G(\mathfrak{o})$ is not 0. A function in the Hecke algebra acts by convolution on this subspace.

There is a well known way to obtain all such representations. Let χ be a character of $T/T(\mathfrak{o}) = A/A(\mathfrak{o})$ with values in $\mathbb{C}^\times$, and let

$$I(\chi) = \{f\colon G \to R \,|\, f(bg) = \delta^{1/2}(b)\chi(b)f(g)\}$$

where

$$\delta^{1/2} : b \longmapsto |\det \mathrm{Ad}(b)|$$

is the modulus character of B. It is there because B is not a unimodular group. Because of the Iwasawa decomposition $G = B \cdot G(\mathfrak{o})$, the subspace of vectors fixed by $G(\mathfrak{o})$ has dimension one. The Hecke algebra acts on it by scalar multiplication, so we get a ring homomorphism φ_χ from $\mathcal{H}$ to $\mathbb{C}$.

Choose a basis (t_i) for the lattice $T/T(\mathfrak{o})$. Characters χ of $\mathcal{T}$ are parametrized by an array (s_i), where $\chi(t_i) = s_i$. Necessarily, each s_i is invertible. For a given f in $\mathcal{H}$, the function taking f to φ_χ as a function of χ is a polynomial in the variables $s_i^{\pm}$. Let R be the ring of all such polynomials. The representations $I(\chi)$ and $I(w\chi)$ are generically isomorophic for w in the Weyl group. We therefore have a map from $\mathcal{H}$ to the ring of invariants R^W. *It is an isomorphism.*

The points (s_i) make up a copy of $(\mathbb{C}^\times)^n$, and can be seen as the complex points on a torus $\widehat{T}$. It may be canonically identified with $\mathrm{Hom}(L^\vee, \mathbb{C}^\times)$. The character group of $\widehat{T}$ is the cocharacter group of T.

This torus sits in turn in a complex reductive group called the L-group.

Let me summarize. If F is a field, T a torus defined over F, and t in $T(F)$, then the map taking an algebraic character λ to $\lambda(t)$ identifies $T(F)$ with $\mathrm{Hom}(X^*(T), F^{|times})$.

An unramified character of T is a homomorphism from the lattice $L^\vee = T/T(\mathfrak{o})$ to $\mathbb{C}^\times$. It may be identified with the complex points on a torus $\widehat{T}$ whose character lattice is $L^\vee$, whereas the character lattice of T is L. Principal series are parametrized by W-orbits on $\widehat{T}$, and also by homomorphisms from $\mathcal{H}\big(G//G(\mathfrak{o})\big)$ to $\mathbb{C}$.

6. The L-group

The complex group dual to G is the reductive group $\widehat{G}$ defined over $\mathbb{C}$ associated to the dual root datum

$$(L^\vee, \Delta^\vee, L, \Delta)\,.$$

The automorphism σ corresponds to an automorphism of this group preserving a Borel subgroup $\widehat{B}$ containing $\widehat{T}$. Thence a semi-direct product ${}^LG = \widehat{G} \rtimes \langle\sigma\rangle$.

The lattice $L^\vee$ is now the character group of a complex torus $\widehat{T}$. Unramified characters of T correspond to elements of $\widehat{T}$. If G is split and σ is trivial, a Weyl group orbit of unramified characters corresponds to a semi-simple conjugacy class in the coset $\widehat{G}$. In the general case, it turns out that they correspond to conjugacy classes in the coset $\widehat{G} \times \sigma$.

Semi-simple conjugacy classes in ${}^LG$ correspond to homomorphisms from $\mathcal{H}$ to $\mathbb{C}$.

A conjugacy class in $\widehat{G} \times \sigma$ corresponds to a homomorphism from $\mathcal{H}$ to $\mathcal{C}$. So does the character of a representation of ${}^LG$.

The integral Hecke algebra may be identified with a ring of certain representations of ${}^LG$.

7. Endoscopic groups

An unramified endoscopic group of G is an unramified reductive group H with datum

$$(L, \Sigma_H, L^\vee, \Sigma_H^\vee, \sigma_H)$$

subject to the condition that there exist s in $\widehat{T} = \mathrm{Hom}(L^\vee, \mathbb{C}^\times)$ with

(a) $\Sigma_H^\vee = \mathrm{Ker}(s) \cap \Sigma_G^\vee$

(b) $\sigma_H = w \cdot \sigma_G$ **for some w in W**

(c) $\sigma_H(s) = s$

The element s is by no means unique —it is not actually part of the data describing an endoscopic group. (Maybe it should be?)

The group ${}^LH$ often embeds into ${}^LG$. When it does, each emi-simpe class in $\widehat{H} \times \sigma$ givs rise to one of $\widehat{G} \times \sigma$, and the map backwards is the restriction of a representation of ${}^LG$ to one of ${}^LH$, or in othr words a map from the Hecke algebra of G to that of H.

But sometimes ${}^LH$ does not embed into ${}^LG$ ...

Example. $G = \mathrm{SL}(2)$. **The group is split, so $\sigma_G = I$.**

The lattices L and $L^\vee$ may be identified with $\mathbb{Z}$, $\alpha^\vee$ with 1, α with 2. Then $\widehat{T} = \mathrm{Hom}(L^\vee, \mathbb{C}^\times)$ is $\mathbb{C}^\times$.

The only s annihilating any roots is $s = 1$. So $\Sigma_H = \emptyset$ if $s \neq 1$ and all of Σ_G otherwise.

If $w = 1$, so is σ_H. (c) makes is no condition on s. So there are two cases: (i) $s = 1$ when $H = G$, and (ii) $s \neq 1$ when H is the split one-dimensional torus.

If $w \neq 1$, then σ_H is non-trivial, and the torus T is the norm-one subgroup of the multiplicative group of the unramified quadratic extension. It must stabilize positive roots if there are any, amd this does not happen, and s must be 1. Condition (c) is vacuous. Therefore H itself is (iii the unramified non-split torus.

Example. $G = \mathrm{PGL}(2)$. The group is again split and $\sigma_G = I$. Here $L^\vee = \mathbb{Z}$ but $\alpha^\vee = 2$. So both $s = \pm 1$ fix all roots.

When $w = 1$ there are again two possibilities, (i) the split torus and (ii) $\mathrm{PGL}(2)$ itself. In the second case, both $s = \pm 1$ will be compatible.

If $w \neq 1$, then (c) tells us that $w(s) = s$, so $s = \pm 1$ and $\Sigma_H = \Sigma_G$. But in this case w does not fix the positive roots, so this case is disallowed.

Example. $G = U(3)$**?**

It would be nice to have a simple algorithm for listing all possible endoscopic groups, along with information about embeddings of ${}^L H$ into ${}^L G$.

It is not s, but at most the W-orbit of s that matters. This orbit corresponds to an unramified representation, and presumably endoscopy says something about its character. What is conjectured and what proven?