An introduction to Hilbert-Siegel modular forms and automorphic representations

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Automorphic Forms Seminar

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Remember the goal of this Seminar: to deepen our understanding of automorphic representations and all the affiliated goodies, like L-functions, Galois representations, and in certain cases, modular forms and abelian varieties.

It is worth keeping the main issues in mind: the Langlands Correspondence and the Principle of Functoriality (which implies the Jacquet-Langlands Correspondence).

The goal of this series of talks is to briefly explore these concepts as they pertain to the group $GSp(2n)$ and an important class of automorphic representations of $GSp(2n)$ - which includes those that correspond to Hilbert-Siegel modular forms.

It is worth noting that $GSp(2) = GL(2)$; only this case is well understood.

Fields

- Throughout this talk we write F for a number field and $\mathbb{A}$ for the ring of adeles for F.
- The subring of finite adeles will be denoted $\hat{F}$ and the maximal order in $\hat{F}$ will be denoted $\hat{\mathcal{O}}_F$.
- We write S_∞ for the set of archimedean places for F.

Groups

Let G be an arbitrary connected reductive algebraic group over F and let Z be the centre of G.

- Set $G_0 := G(\hat{F})$, $K_0 = G(\mathfrak{o}_{\hat{F}})$ and $Z_0 := Z(\hat{F})$.
- Likewise, define $G_\infty := \prod_{v \in S_\infty} G(F_v)$ and $Z_\infty := \prod_{v \in S_\infty} Z(F_v)$.
- Let K_∞ be a fixed maximal compact subgroup of G_∞.
- Let $\omega : Z(\mathbb{A}) \to \mathbb{C}$ be a fixed continuous unitary character, let $\omega_0 : Z_0 \to \mathbb{C}$ denote the restriction of ω to Z_0 and let $\omega_\infty : Z_\infty \to \mathbb{C}$ denote the restriction of ω to Z_∞.
- We have in mind: $Z \cong GL(1)$.

The Hecke algebra over finite adeles

Let $\mathcal{H}_0 = \mathcal{H}(G_0, \omega_0)$ denote the convolution algebra of locally constant functions $f : G_0 \to \mathbb{C}$ such that the support is compact modulo Z_0 and $f(\mathfrak{z}g) = \omega_0^{-1}(\mathfrak{z})f(g)$ for all $\mathfrak{z} \in Z_0$ and $g \in G_0$.

- In order to define convolution we must pick fix Haar measure on G_0, which we do by insisting that the measure of K_0 is 1. With this choice, convolution is given by

$$(f * g)(x) := \int_{G_0/Z_0} f(y)g(y^{-1}x)\, dy.$$

- Every idempotent in $\mathcal{H}_0$ is a finite linear combination of $e_U := \omega_0^{-1} m(U)^{-1} 1_U$, where U is a compact-mod-centre open subgroup of G_0 and where $m(U) = \int_{G_0/Z_0} 1_U(y)\, dy$.

The Hecke algebra over archimedean places

Let $\mathcal{H}_\infty = \mathcal{H}(G_\infty, K_\infty, \omega_\infty)$ denote the convolution algebra of left-K_∞-finite complex distributions on G_∞ supported by K_∞ modulo Z_∞ such that the restriction of the measure to Z_∞ is ω_∞^{-1}.

- In terms of generalized functions, convolution is given by

$$(\mu_1 * \mu_2)(x) := \int_{G_\infty/Z_\infty} \mu_1(y)\mu_2(y^{-1}x)\, dy,$$

  where dy is the quotient measure on G_∞/Z_∞ obtained from Haar measure on G_∞ normalized with respect to K_∞ and the Haar measure on Z_∞ normalized with respect to $Z_\infty \cap K_\infty$.
- Every idempotent in $\mathcal{H}_\infty$ is a finite linear combination of measures of the form $\omega_\infty^{-1} \frac{\operatorname{trace}\sigma}{\dim \sigma}\, dk\, dz$, where σ is an irreducible representation of K_∞, dk is the normalized Haar measure on K_∞ and dz is the Haar measure on Z_∞ normalized as above.

The full Hecke algebra

Finally, let $\mathcal{H} = \mathcal{H}(G(\mathbb{A}), \omega)$ denote the tensor product $\mathcal{H}_0 \otimes_{\mathbb{C}} \mathcal{H}_\infty$; this is the *Hecke algebra* for $G(\mathbb{A})$ (with respect to ω^{-1}).

- An idempotent $\xi \in \mathcal{H}$ will be called an *elementary idempotent* if $\xi = \xi_0 \otimes \xi_\infty$ (so ξ is pure) with $\xi_0 = \omega_0^{-1} e_U$ for some open subgroup U of G_0 compact modulo Z_0 and $\xi_\infty = \omega_\infty^{-1} \frac{\mathrm{trace}\sigma}{\dim \sigma}\, dk\, dz$ for some irreducible representation σ of K_∞; in this case we refer to U as the *level* of ξ and σ as the *weight* of ξ.
- If ξ is an idempotent then $\mathcal{H}(\xi) := \xi * \mathcal{H} * \xi$ is a submodule of $\mathcal{H}$. Although $\mathcal{H}$ does not typically have an identity, $\mathcal{H}(\xi)$ certainly does: ξ itself.

Hilbert space

- Every automorphic representation is an irreducible Hecke-module, but not every irreducible Hecke-module is an automorpic representation.
- In order to define automorphic representations (as they appear in this talk) we begin with the most important Hecke-module, which is the Hilbert space
$$L^2(G(F)\backslash G(\mathbb{A}), \omega)$$
of square-integrable functions $\phi : G(\mathbb{A}) \to \mathbb{C}$ such that
$$\forall \mathfrak{z} \in Z(\mathbb{A}), \forall \gamma \in G(F), \forall g \in G(\mathbb{A}), \quad \phi(\mathfrak{z}\gamma g) = \omega(\mathfrak{z})\phi(g).$$

Hilbert space Hecke module

- The group $G(\mathbb{A})$ acts on $L^2(G(F)\backslash G(\mathbb{A}), \omega)$ by the right regular representation R: for all $\phi \in L^2(G(F)\backslash G(\mathbb{A}), \omega)$ and for all $x, y \in G(\mathbb{A})$,
$$(R(y)\phi)(x) := \phi(xy).$$
- The action of $\mathcal{H}$ on $L^2(G(F)\backslash G(\mathbb{A}), \omega)$ corresponds to this representation: for $\xi \in \mathcal{H}$ and $\phi \in L^2(G(F)\backslash G(\mathbb{A}), \omega)$ define $R(\xi)\phi : G(\mathbb{A}) \to \mathbb{C}$ by
$$R(\xi)\phi := \int_{G(\mathbb{A})/Z(\mathbb{A})} \xi(y)\ R(y)\phi\, dy.$$

Hilbert space Hecke module

- An elementary calculation shows that

$$R(\xi_1)(R(\xi_2)\phi) = R(\xi_1 * \xi_2)\phi,$$

  for all $\xi_1, \xi_2 \in \mathcal{H}$ and for all $\phi \in L^2(G(F)\backslash G(\mathbb{A}), \omega)$.

Thus, $L^2(G(F)\backslash G(\mathbb{A}), \omega)$ is a $\mathcal{H}$-module.

Invariant differential operators

Let $\mathfrak{g}_\infty$ denote the Lie algebra of the (Real) Lie group G_∞.

- Then $\mathfrak{g}_\infty$ is the cotangent bundle of G_∞:
$$(Xf)(x) := \frac{d}{dt} f(x \exp(tX))\Big|_{t=0}$$
- Let $\mathfrak{Z}$ denote the centre of the enveloping algebra of the complexification $\mathfrak{g}_\infty \otimes_{\mathbb{R}} \mathbb{C}$ of the Lie algebra $\mathfrak{g}_\infty$. Then $\mathfrak{Z}$ is the bundle of left- and right-invariant differential operators on the space of smooth complex-valued functions $G_\infty \to \mathbb{C}$.
- We say $\phi : G(\mathbb{A}) \to \mathbb{C}$ is *smooth* if, for all $c \in G_0$, $\phi_c : G_\infty \to \mathbb{C}$ is smooth, where $\phi_c(x) := \phi(cx)$. If ϕ is smooth and $\mathfrak{Z}$, then
$$X\phi = 0$$
means
$$\forall c \in G_0, \qquad X\phi_c = 0.$$

Automorphic forms

Let J be an ideal of finite co-dimension in $\mathfrak{Z}$.

Definition

Let

$$\mathcal{A}(G_F, J, \omega)$$

denote the space of functions $\phi : G(\mathbb{A}) \to \mathbb{C}$ such that

(af1) ϕ is smooth and slowly increasing

(af2) $\forall X \in J, \quad X\phi = 0$

(af3) $\forall \gamma \in G(F), \forall \mathfrak{z} \in Z(\mathbb{A}), \forall g \in G(\mathbb{A}), \quad \phi(\mathfrak{z}\gamma g) = \omega(z)\phi(g)$

(af4) $\exists$ elementary idempotent $\xi \in \mathcal{H}, \quad R(\xi)\phi = \phi$

Elements of $\mathcal{A}(G_F, J, \omega)$ are called *automorphic forms* (for G_F, J and ω).

Here, the term *slowly increasing* is used in the sense of [BJ1979, §1.2].

Cuspidal automorphic forms

Definition

An automorphic form is *cuspidal* if it further satisfies condition

(af0) $\forall P, \forall g \in L_P(\mathbb{A})$,

$$\int_{U_P(F)\backslash U_P(\mathbb{A})} \phi(ug)\, du = 0,$$

where P ranges over all parabolic subgroups of G defined over F for which the unipotent radical U_P and the reductive quotient L_P are defined over F. The space of cuspidal automorphic forms (with central character ω and type J) is denoted $\mathcal{A}_\circ(G_F, J, \omega)$.

An important Hecke algebra

Proposition

$\mathcal{A}(G_F, J, \omega)$ (resp. $\mathcal{A}_\circ(G_F, J, \omega)$) is a Hecke sub-module of $L^2(G(F)\backslash G(\mathbb{A}), \omega)$ (resp. $L^2_\circ(G(F)\backslash G(\mathbb{A}), \omega)$).

An important restricted Hecke algebra

Definition

For each $\xi \in \mathcal{H}(G(\mathbb{A}_F), \omega)$, we write $\mathcal{A}(G_F, J, \xi, \omega)$ (resp. $\mathcal{A}_\circ(G_F, J, \xi, \omega)$) for the space of $\phi \in \mathcal{A}(G_F, J, \omega)$ (resp. $\phi \in \mathcal{A}_\circ(G_F, J, \omega)$) such that $R(\xi)\phi = \phi$.

Proposition

$\mathcal{A}(G_F, J, \xi, \omega)$ and $\mathcal{A}_\circ(G_F, J, \xi, \omega)$ are $\mathcal{H}(\xi)$-modules

An important spectrum

- Let S_ξ denote the complement of the set of non-archimedean places v of F such that $\mathrm{supp}(\xi_v)$ is a hyperspecial maximal commpact-mod-centre open subgroup of $G(F_v)$.
- Let $\mathcal{H}_{S_\xi}$ denote the restricted direct product, taken over all $v \notin S_\xi$, of the spherical Hecke algebras $\mathcal{H}(\xi)_v$.
- Then $\mathcal{H}(\xi)$ is a semi-simple algebra, so $\mathcal{A}(G, J, \xi)$ admits a basis of $\mathcal{H}_{S_\xi}$-eigenvectors.

Automorphic representations

Definition

An *automorphic representation* π of $G(\mathbb{A})$ is an irreducible Hecke-module which is equivalent to a subquotient of $\mathcal{A}(G_F, J, \omega)$ (with the action R above) for some ideal J in $\mathfrak{Z}$ with finite co-dimension.

Exercise: π is cuspidal (as a representation) if and only if ϕ is cuspidal (as an automorphic form).

Automorphic representations and automorphic forms

Let π be an $\mathcal{H}$-submodule of $\mathcal{A}(G_F, J)$.

- It follows from the definition of $\mathcal{A}(G_F, J)$ that there is some idempotent $\xi \in \mathcal{H}$ such that the vector space $\pi^\xi := \{\phi \in \pi \mid R(\xi)\phi = \phi\}$ is non-trivial. Suppose ξ is elementary idempotent and maximal with this property.
- Now, consider the decomposition $\pi = \otimes_v \pi_v$ where v runs over the set of places for F. Let S_π denote compliment of the set of non-archimedean places v for F such that $\operatorname{supp}(\xi_v)$ is a hyperspecial maximal compact open subgroup of G_v and $\dim_{\mathbb{C}} \pi_v^{\xi_v} = 1$.
- The set S_π finite.
- For each $v \notin S$, the convolution algebra $\mathcal{H}(\xi)_v$ is semi-simple and isomorphic to the spherical Hecke algebra $\mathcal{H}(G(F_v), K_v)$.
- Let $\mathcal{H}_{S_\pi}$ denote the restricted direct product, taken over all $v \notin S_\pi$, of the spherical Hecke algebras $\mathcal{H}(\xi)_v$. Then $\mathcal{H}_{S_\pi}$ is a semi-simple algebra over $\mathbb{C}$.

Cuspidality...

Now, suppose π is a cuspidal automorphic representation.

- Thus, there is some ideal J such that π is (isomorphic to) a summand of $\mathcal{A}(G_F, J)$.
- Let ξ be a maximal elementary idempotent such that π^ξ is non-trivial.
- Then π^ξ is a $\mathcal{H}(\xi)$-submodule of $\mathcal{A}(G, J, \xi)$.
- Since $\mathcal{H}_{S_\pi}$ is a subalgebra of $\mathcal{H}(\xi)$, it follows that π^ξ is a $\mathcal{H}_{S_\pi}$-module. Since $\mathcal{H}_{S_\pi}$ is a semi-simple algebra, π^ξ admits a basis of $\mathcal{H}_{S_\pi}$-eigenvectors.
- Let $\phi \in \pi$ be a $\mathcal{H}_{S_\pi}$-eigenvector. Since $\mathcal{H}\phi$ is a non-trivial $\mathcal{H}$-invariant subspace of π and since π is irreducible, it follows that $\pi = \mathcal{H}\phi$.
- In this way we see that the cuspidal automorphic representation π is generated by a $\mathcal{H}_{S_\pi}$-eigenform in $\mathcal{A}_\circ(G, J, \xi)$.
- Let $\phi \in \mathcal{A}(G, J, \xi)$ be a $\mathcal{H}_{S_\xi}$-eigenvector. Up to equivalence, $\mathcal{H}\phi$ contains exactly one automorphic representation π_ϕ.

The (partial) L-function

(Suppose, for each place v, that G_{F_v} splits over a cyclic extension; for each place v we fix an element $\tau \in WD_{F_v}$ such that the twist of G_{F_v} by τ is split.)
Let $r : {}^L G(\mathbb{C}) \to GL(m, \mathbb{C})$ be a representation and let π be a $\mathcal{H}$-submodule of $\mathcal{A}(G, J)$.

- Let ξ be a maximal elementary idempotent such that π^ξ is non-trivial.
- Let S_π be the finite set of valuations defined above.
- Now, the action of $\mathcal{H}(\xi)_v$ on V_v is given by a character, which by the Satake transform, defines a semi-simple element z_v in ${}^L G^\circ \times \tau$.
- For each $v \notin S_\pi$, define

$$L_v(s, \pi, r) := \det \left(1 - q^{-s} r(z_v)\right)^{-1},$$

and

$$L_{S_\pi}(s, \pi, r) := \prod_{v \notin S_\pi} L_v(s, \pi, r).$$

Vector-valued automorphic forms

Let $\sigma : K_\infty \to \mathrm{Aut}_{\mathbb{C}}(W)$ be an irreducible representation and let U be a compact open subgroup of G_0. Let

$$\mathcal{A}(G_F, W, U, J)$$

denote the set of functions $\Phi : G(\mathbb{A}_F) \to W$ such that

(AF1) Φ is smooth and slowly increasing

(AF2) $J\Phi = 0$

(AF3) $\forall \mathfrak{z} \in Z(\mathbb{A}_F), \forall \gamma \in G(F), \forall g \in G(\mathbb{A}), \forall k_0 \in U$,

$$\Phi(\mathfrak{z}\gamma g k_0) = \omega(z)\Phi(g)$$

(AF4) $\forall k_\infty \in K_\infty, \forall g \in G(\mathbb{A}_F)$,

$$\Phi(g k_\infty) = \sigma(k_\infty^{-1})\Phi(g)$$

Elements of $\mathcal{A}_{\mathbb{C}}(G_F, W, U, J)$ are called *W-valued automorphic forms* of weight σ, level U and central character ω.

Vector-valued cuspidal automorphic forms

A W-valued automorphic form is *cuspidal* if it further satisfies condition

(AF0) $\forall P, \forall g \in L_P(\mathbb{A})$,

$$\int_{U_P(F)\backslash U_P(\mathbb{A})} \Phi(ug)\, du = 0,$$

where P ranges over all parabolic subgroups of G defined over F for which the unipotent radical U_P and the reductive quotient L_P are defined over F. The set of cuspidal W-valued automorphic forms on $G(\mathbb{A})$ of weight σ and level U with central character ω is denoted

$$\mathcal{A}_\circ(G_F, W, U, J).$$

Scalar *vs.* vector-valued automorphic forms

Proposition

Let U be a compact open subgroup of G_0, let $\sigma : K_\infty \longrightarrow \mathrm{Aut}_{\mathbb{C}}(W)$ be an irreducible unitary representation. Let ξ be the elementary idempotent in the Hecke algebra $\mathcal{H}$ corresponding to σ and U. Then $\mathcal{A}(G_F, W, U, J)$ and $\mathcal{A}(G_F, \xi, J)$ are isomorphic Hecke-modules; likewise, $\mathcal{A}_\circ(G_F, W, U, J)$ and $\mathcal{A}_\circ(G_F, \xi, J)$ are isomorphic Hecke-modules.

Proof:

For every non-zero $w \in W$ and $\Phi \in \mathcal{A}(G_F, W, U, J)$, define

$$\langle \Phi | w \rangle : G(\mathbb{A}) \to \mathbb{C}$$

by

$$\forall g \in G, \quad \langle \Phi | w \rangle (g) := \langle \Phi(g) | w \rangle.$$

We claim that

$$\begin{aligned} \mathcal{A}(G_F, W, U, J) &\to \mathcal{A}(G_F, \xi, J) \\ \Phi &\to \langle \Phi | w \rangle \end{aligned}$$

is a Hecke-module isomorphism.

Symplectic linear transformations

Let $Sp(2n)$ denote the group scheme

$$Sp(2n) = \left\{ g \in GL(2n) \mid {}^t g J_n g = J_n \right\},$$

where

$$J_n = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}.$$

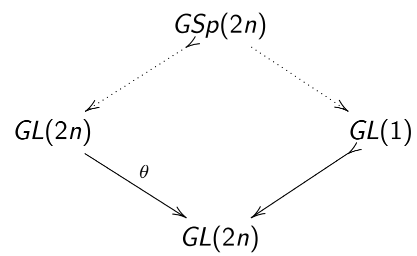
- Observe that $Sp(2n)$ is the subgroup of $GL(2n)$ corresponding to the elements fixed by the involution

$$\begin{aligned} \theta : GL(2n) &\to GL(2n) \\ g &\mapsto {}^t J_n \, {}^t g J_n g. \end{aligned}$$

- The centre of $Sp(2n)$ is NOT $GL(1)$.

Symplectic similitudes

$GSp(2n)$ is the fibred product of $GL(2n)$ with $GL(1)$ using this involution:



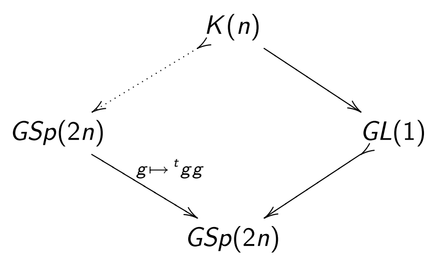
- Thus, we may view $GSp(2n)$ as a group subscheme in $GL(2n)$ and write

$$GSp(2n) = \left\{ g \in GL(2n) \mid \exists \nu \in GL(1),\ {}^tJ_n g J_n = \nu 1_{2n} \right\}.$$

- The centre of $GSp(2n)$ is $GL(1)$.

Another very important group scheme

Definition
Let $K(n)$ denote the group scheme obtained by pulling back ...



- Thus, $K(n) = GSp(2n) \cap GO(2n)$.

Matrix forms

For future reference:

- $$GSp(2n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2n) \middle| \begin{matrix} {}^t ad - {}^t cb \in Z(n) \\ {}^t ac = {}^t ca \\ {}^t bd = {}^t db \end{matrix} \right\}$$
- $$K(2n) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in GL(2n) \middle| \begin{matrix} {}^t aa + {}^t bb \in Z(n) \\ {}^t ab = {}^t ba \end{matrix} \right\}.$$

where $Z(n)$ is the centre of $GL(n)$.

Hilbert-Siegel automorphic representations

A *Hilbert-Siegel automorphic representation* is a complex automorphic representation isomorphic to a subquotient of $\mathcal{A}(G_F, J)$ where

- F is a totally real number field,
- $G_F = GSp(2n)_F$, and
- J is the ideal in $\mathfrak{Z}$ generated by

$$\mathfrak{h} := \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \in \mathfrak{g}_\infty \otimes \mathbb{C} \middle| a - bi = 0 \right\}.$$

- the maximal compact subgroup K_∞ appearing in the definition of the Hecke algebra $\mathcal{H}_\infty$ is

$$\prod_{v \in S_\infty} K(F_v).$$

Observe that Hilbert-Siegel automorphic representations take arbitrary weight U, level σ and central character ω.

$K(n, \mathbb{R})$

The class of Hilbert-Siegel automorphic representations *includes* those representations corresponding to classical Hilbert-Siegel modular forms. To see this, we must study $K(n) \longrightarrow GSp(2)$ over $\mathbb{R}$.

Lemma

$$i_{\mathbb{R}} : K(n, \mathbb{R}) \longrightarrow GU(n, \mathbb{C})$$
$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto a - ib$$

is an isomorphism.

$K(n,\mathbb{R}) \hookrightarrow GSp(2n,\mathbb{R})$

Proposition

In the category of smooth real manifolds, the map

$$K(n,\mathbb{R}) \hookrightarrow GSp(2n,\mathbb{R})$$

admits an $n(n+1)$-dimensional cokernel; moreover, this cokernel carries a complex structure.

Torsor:

$$K(n,\mathbb{R}) \rightarrowtail GSp(2n,\mathbb{R}) \twoheadrightarrow GSp(2n,\mathbb{R})/K(n,\mathbb{R})$$

The complex structure for $GSp(2n,\mathbb{R})/K(n,\mathbb{R})$

Proposition

$$GSp(2n,\mathbb{R}) \to GL(n,\mathbb{C})$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to (ai+b)(ci+d)^{-1}.$$

- *The image of this function is the set of complex, symmetric, invertible $n \times n$-matrices with either positive-definite or negative-definite imaginary part.*
- *The pre-image of $i1_n$ is $K(n,\mathbb{R})$.*
- *We may identify $GSp(2n,\mathbb{R})/K(n,\mathbb{R})$ with the image of the map above.*

Hilbert-Siegel space

Definition

$$\Omega(u) := G_\infty / K_\infty$$

Then,

$$\Omega(n) \cong (-\mathbb{H}_n \cup \mathbb{H}_n)^d,$$

where $d = \#S_\infty$.

Observe that G_∞ acts on $\Omega(n)$.

Example: $n = 1,\ F = \mathbb{Q}$

- $$K(1,\mathbb{R}) \cong GU(1,\mathbb{C}) \cong GO(2,\mathbb{R}).$$
- $$\Omega(1) = GL(2,\mathbb{R})/GO(2,\mathbb{R}) = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R}).$$
- Aside: Compare this with the Drinfeld space (over $\mathbb{Q}_p$):
$$\Omega^1_{\mathbb{Q}_p} = \mathbb{P}^1(\bar{\mathbb{Q}}_p) - \mathbb{P}^1(\mathbb{Q}_p).$$
- The connected component of $\Omega(1)$ is the 'upper-half' of the complex plane.

Off on a tangent ...

$$K_\infty \longrightarrow G_\infty \xrightarrow{f} \Omega(n)$$

$$\mathfrak{k}_\infty \longrightarrow \mathfrak{g}_\infty \xrightarrow{df} T_I\Omega(n) \xrightarrow{\cong} T_I\mathbb{H}_n^d$$

$$\mathfrak{k}_\infty \otimes \mathbb{C} \longrightarrow \mathfrak{g}_\infty \otimes \mathbb{C} \xrightarrow{df} T_I\Omega(n) \otimes \mathbb{C} \xrightarrow{\cong} \mathrm{sym}(n, \mathbb{C})^d$$

and

$$\mathfrak{g}_\infty \otimes \mathbb{C}/\mathfrak{k}_\infty \otimes \mathbb{C} \cong \mathfrak{p}_\mathbb{C} := \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \middle| \, {}^t a = a, \, {}^t b = b \right\}$$

Complex structure on $T_I\Omega(n) \otimes \mathbb{C}$

The complex structure on $\Omega(n)$ determines an anti-idempotent J on $T_I\Omega(n) \otimes \mathbb{C} \cong \mathfrak{p}_{\mathbb{C}}$:

$$J \begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \begin{pmatrix} b & -a \\ -a & -b \end{pmatrix}$$

Decompose by $\pm i$-eigenspaces and use the isomorphism $\mathfrak{p}_{\mathbb{C}} \cong \mathrm{sym}(n, \mathbb{C})^d$:

$$\mathfrak{p}_{\mathbb{C}}^+ = \left\{ \begin{pmatrix} a & +ia \\ +ia & -a \end{pmatrix} \middle| {}^t a = a \right\} \cong \mathrm{span}_{\mathbb{C}} \left\{ \frac{d}{dz_j} \middle| j \right\}$$

$$\mathfrak{p}_{\mathbb{C}}^- = \left\{ \begin{pmatrix} a & -ia \\ -ia & -a \end{pmatrix} \middle| {}^t a = a \right\} \cong \mathrm{span}_{\mathbb{C}} \left\{ \frac{d}{d\bar{z}_j} \middle| j \right\}$$

Automorphy 'factor'

(Henceforth, for simplicity, restrict to the case of trivial central character.)

- Let $\sigma : K_\infty \to \mathrm{Aut}_{\mathbb{C}}(W)$ be an irreducible unitary representation.
- Let $\sigma : \prod_{v \in S_\infty} GL(n, \mathbb{C}) \to \mathrm{Aut}_{\mathbb{C}}(W)$ be the irreducible unitary representation with weight vector matching that of σ and trivial central character.
- Define
$$\begin{aligned} j_\sigma : G_\infty \times \Omega(n) \to & \quad \mathrm{Aut}_{\mathbb{C}}(W) \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) \mapsto & \quad \sigma(cz + d) \end{aligned}$$
This is called the weight-σ *automorphy factor*.

Properties of the automorphy 'factor'

- If $k \in K_\infty$ then
$$j_\sigma(k, I) = \sigma(k).$$
- Cocycle relation: $\forall g, h \in GSp(2n, \mathbb{R}),\ \forall z \in \Omega(n)$,
$$j_\sigma(gh, z) = j_\sigma(g, hz) \circ j_\sigma(h, z).$$

Hilbert-Siegel Modular forms

Let Γ be an arithmetic subgroup of G_∞ and let $\sigma : K_\infty \to \mathrm{Aut}_{\mathbb{C}}(W)$ be an irreducible unitary representation. Let $M_\sigma(GSp(2n)_F, \Gamma)$ denote the space of functions

$$F : \Omega(n) \to W$$

such that

(MF1) $\forall \gamma \in \Gamma, \forall z \in \Omega(n)$,

$$F(\gamma z) = j_\sigma(\gamma, z) F(z),$$

(MF2) F is holomorphic.

From automorphic forms to modular forms

- Let U be a compact open subgroup of G_0 and let C be a set of representatives for the finite double quotient space

$$GSp(2n, F)\backslash GSp(2n, \mathbb{A}_F)/U.$$

- Let $\sigma : K_\infty \to \mathrm{Aut}_{\mathbb{C}}(W)$ be an irreducible unitary representation.
- For each $c \in C$, define

$$\Gamma_c = GSp(2n, F) \cap (G_\infty \times cUc^{-1}).$$

  The image of Γ_C under $GSp(2n, F) \to G_\infty$ is an arithmetic subgroup of G_∞.
- For each $\Phi \in \mathcal{A}(GSp(2n)_F, U, W, J)$ define $F_{\Phi_c} : \Omega(n) \to W$ by: $\forall z \in \Omega, \exists g \in G_\infty$,

$$z = g \cdot I \quad \text{and} \quad F_{\Phi_c}(z) = j_\sigma(g, I)\Phi(cg).$$

... is well-defined:

Suppose $z = g \cdot I = g' \cdot I$. Then $g' = gk$ for some $k \in K_\infty$:

$$\begin{aligned} j_\sigma(g', I)\Phi(g') &= j_\sigma(gk, I)\Phi(gk) \\ &= j_\sigma(g, k \cdot I) j_\sigma(k, I)\sigma(k^{-1})\Phi(g) \\ &= j_\sigma(g, I)\sigma(k)\sigma(k^{-1})\Phi(g) \\ &= j_\sigma(g, I)\Phi(g) \end{aligned}$$

Thus, F_Φ is well-defined.

Theorem

With U, σ, ω and C as above,

$$\mathcal{A}(GSp(2n)_F, U, W, J) \longrightarrow \oplus_{c\in C} M_\sigma(GSp(2n)_F, \Gamma_c)$$
$$\Phi \mapsto (F_{\Phi_c})_{c\in C}$$

is an isomorphism is Hecke-modules. Furthermore, this isomorphism restricts to a Hecke-module isomorphism from cuspidal automorphic forms to cuspidal modular forms.

Equivariance

$\forall\gamma \in \Gamma, \forall z \in \Omega(n), \exists g \in G_\infty$:

$$
\begin{aligned}
z = g\cdot I \quad \text{and} \quad & F_\Phi(\gamma\cdot z)\\
&= F_\Phi(\gamma\cdot g\cdot I)\\
&= F_\Phi((\gamma g)\cdot I)\\
&= j_\sigma(\gamma g, I)\Phi(\gamma g)\\
&= j_\sigma(\gamma, g\cdot I) j_\sigma(g, I)\Phi(g)\\
&= j_\sigma(\gamma, z) F_\Phi(g)
\end{aligned}
$$

From modular forms to automorphic forms

With notation as above, suppose $F \in M_\sigma(GSp(2n)_F, \Gamma)$ and define $\Phi_F : GSp(2n, \mathbb{A}_F) \to W$ by:

$\forall g \in G(\mathbb{A}_F), \exists g_F \in G(F), \exists g_\infty \in G_\infty, \exists k_0 \in K_0,$

$$g = g_F g_\infty k_0 \quad \text{and} \quad \Phi_F(g) = j_\sigma(g_\infty, I)^{-1} F(g_\infty \cdot I).$$

Equivariance

$\forall k_\infty \in K_\infty, \forall g \in GSp(2n, \mathbb{A}_F), \exists z \in \Omega(n),$

$$
\begin{aligned}
z = g_\infty \cdot I \quad \text{and} \quad & \Phi_F(g k_\infty) \\
& = j_\sigma(g k_\infty, I)^{-1} F(g k_\infty \cdot I) \\
& = j_\sigma(k_\infty, I)^{-1} j_\sigma(g, k_\infty \cdot I)^{-1} F(g \cdot k_\infty \cdot I) \\
& = \sigma(k_\infty)^{-1} j_\sigma(g, I)^{-1} F(g \cdot I) \\
& = \sigma(k_\infty)^{-1} \Phi_F(g)
\end{aligned}
$$

Holomorphy

Lemma

With notation above, if $F = F_\Phi$ or $\Phi = \Phi_F$, then F is holomorphic if and only if $\mathfrak{p}_{\mathbb{C}}^{-}\Phi = 0$.

Define $j_\infty : G_\infty \to \mathrm{Aut}_{\mathbb{C}}(W)$, $F_\infty : G_\infty \to W$ and $\Phi_\infty : G_\infty \to W$ by

$$\begin{aligned} j_\infty(g) &:= j_\sigma(g, I)^{-1} \\ F_\infty(g) &:= F(g_\infty \cdot I) \\ \Phi_\infty(g) &:= j_\infty(g) F_\infty(g) \end{aligned}$$

From work above it follows that that j_∞ is holomorphic (so $\mathfrak{p}_{\mathbb{C}}^{-} j_\infty = 0$) and that F_∞ is holomorphic if and only if $\mathfrak{p}_{\mathbb{C}}^{-} F_\infty = 0$.