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The Weil Representation

References

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$$K = \text{ a field.}$$

Eventually, $K =$ the reals, a local field, or a finite field, of odd characteristic

$$V = \text{ a vector space over } K, \quad \dim V = 2n.$$

$$\langle\ ,\ \rangle = \text{ a non-degenerate alternating } K\text{-bilinear form on } V$$

$$Sp = Sp(V) = Sp(2n, K), \text{ the symplectic group}$$
$$= \{\, g \in GL(V) : \langle gv, gw\rangle = \langle v, w\rangle \text{ for all } v, w \in V\}.$$

$$H = H(V) = \text{Heisenberg group} = K \times V$$
$$(a, v)(b, w) = (a + b + \langle v, w\rangle\,, v + w), \qquad a, b \in K,\ v, w \in V$$

Centre $Z(H)$ of $H = \{\, (c, 0) : c \in K\,\}$.

$$\lambda : K^+ \to \mathbb{C}^*, \text{ a character of } K^+$$

In the classical real case, $K = \mathbb{R}$, $\lambda(x) = e^{2\pi ix}$; the Stone-von Neumann Theorem says that there exists an irreducible unitary representation of H, called the Schrödinger representation, such that $(c, 0)$ acts as the scalar $\lambda(c)$, and this representation is unique up to equivalence. There are versions of this for our other fields.

The Schrödinger model

Suppose that V has basis $x_1, x_2, \dots, x_n, y_n, \dots, y_2, y_1$ such that

$$\langle x_i, y_j \rangle = \delta_{ij}$$

$$X = \text{ span of } x_1, x_2, \dots, x_n$$
$$Y = \text{ span of } y_1, y_2, \dots, y_n$$

$$V = X \oplus Y$$
$$\langle x, x' \rangle = 0,\ x, x' \in X \quad \langle y, y' \rangle = 0,\ y, y' \in Y$$
$$A = \{\, (c, y) \mid y \in Y \,\}$$

so A is abelian.
Regard λ as a character of $(K, 0) = Z(H)$, and extend λ to A by defining $\lambda(c, y) = \lambda(c)$, $c \in K$, $y \in Y$.
Let S be the representation of H given by $\mathcal{I} = \operatorname{ind}_A^H \lambda$.
This is defined to be the space of functions $f : H \to \mathbb{C}$ such that

$$f(ah) = \lambda(a) f(h), \qquad \text{for all } a \in A, \quad h \in H.$$

The group H acts on these functions by right translation:

$$S(h) f(h') = f(h'h).$$

For $(c, 0) \in Z(H)$,

$$S(c, 0) f(h) = f\big(h(c, 0)\big) = f\big((c, 0)h\big) = \lambda(c) f(h)$$

so $Z(H)$ does indeed act as multiplication by λ.
One way to construct these functions in $\mathcal{I}$ is as follows.

Pick coset representatives $\mathcal{T}$ for A in H.

Suppose that $\phi : \mathcal{T} \to \mathbb{C}$ is a function.

For $h \in H$, write $h = at$ where $t \in \mathcal{T}$, $a \in A$. Define f on H by defining

$$f(h) = f(at) = \lambda(a)\phi(t).$$

Then check that $f(ah) = \lambda(a)f(h)$. So $\mathcal{I} = \operatorname{ind}_A^H \lambda$ is given by functions on $\mathcal{T}$.
In our case, $A = \{ (c, y) \mid y \in Y \}$, and since $V = X \oplus Y$, then coset representatives of A in H are given by

$$\mathcal{T} = \{ (0, x) = t_x \mid x \in X \}$$

So the underlying space of our model $\mathcal{I}$ can be identified with functions $\mathcal{S}(X)$ on X.

$$\phi(x) = f\big((0, x)\big).$$

How does $S(h)$ act on ϕ? Suppose that $h = (c, x + y)$, where $c \in K$, $x \in X$, $y \in Y$.

$$\begin{aligned} S(c, x + y)\phi(x') &= f\big((0, x')(c, x + y)\big) \\ &= f\big((c + \langle x', y \rangle, x' + x + y)\big) \\ &= f\big((c + 2\langle x', y \rangle + \langle x, y \rangle, y)(0, x + x')\big) \\ &= \lambda(c + 2\langle x', y \rangle + \langle x, y \rangle)\phi(x' + x) \end{aligned}$$

One takes $\mathcal{S}(X)$ to be the set of all (complex-values) L_2-functions on $X = K^n$, giving us a Hilbert space.

In the real case, $X = \mathbb{R}^n$, and one uses usual Lebesgue measure.

In the finite case, one uses counting measure, so $\mathcal{S}(X)$ is the set of all functions on X.
In the p-adic case, one uses a Haar measure on K.
In the finite case, Stone-von Neumann follows from the fact that the character χ of S satisfies

$$\chi(c, v) = 0 \quad \text{if } v \neq 0.$$

It then follows that the character inner product $(\chi, \chi) = 1$, so χ is irreducible, and if ζ is the character of another irreducible representation in which $Z(H)$ acts via λ then $(\chi, \zeta) = 1$ so $\chi = \zeta$. If K has q elements, then X has q^n elements, so

$$\dim \mathcal{S}(X) = q^n.$$

There is an action of the symplectic group $Sp(V) = Sp$ on H:

$$g(a,v) = (a, gv),\ a \in K,\ v \in V,\ g \in Sp.$$

Sp acts trivially on $Z(H)$.

Define ${}^gS(h) = S(gh)$, for $g \in Sp,\ h \in H$.

Then ${}^gS$ is an irreducible representation of H, and since Sp acts trivially on $Z(H)$, then for $(c,0) \in Z(H)$, ${}^gS(c,0)$ acts as $\lambda(c)$.
Then Stone-von Neumann implies that ${}^gS$ is equivalent to S. So there is an operator $W(g)$ on the representation space of S such that

$$W(g)S(h)W(g)^{-1} = {}^gS(h) = S(gh),$$

$$g \in Sp(V), \quad h \in H.$$

From Schur's Lemma, $W(g)$ is unique up to a scalar multiple.

W is called the Weil or oscillator or metaplectic representation.
From Schur's Lemma,

$$W(g_1g_2) = \alpha(g_1,g_2)W(g_1)W(g_2) \text{ for some } \alpha(g_1,g_2) \in \mathbb{C}^*, g_1, g_2 \in Sp$$

W is called a projective representation of $Sp(V)$; α is a 2-cocyle.
Recall that $V = X \oplus Y$. Define

$$P = \{\, g \in Sp(V) : gY = Y \,\}.$$

Then P is a maximal parabolic subgroup of $G = Sp$. Using the basis $x_1, x_2, \ldots, y_n$, elements of P have the form

$$\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$$

Then $A = K \times Y$ is invariant under the action of P, so we can extend the character λ to the semi-direct product $A \rtimes P$, and form the induced representation

$$\mathrm{ind}_{A\rtimes P}^{H\rtimes P} \lambda.$$

It follows that it is easy to determine W on P, and the cocycle is trivial on P.
The space $\mathcal{S}(X)$ is the direct sum of even functions and odd functions,

$$\mathcal{S}(X) = \mathcal{S}(X)_e + \mathcal{S}(X)_o.$$

We claim that each of $\mathcal{S}(X)_e$ and $\mathcal{S}(X)_o$ is invariant under all $W(g)$, $g \in G$. Let $\iota = -I \in Sp(V)$, so ι is a central involution. For $\phi \in \mathcal{S}(X)$ we have

$$W(\iota)\phi(x) = \phi(-x).$$

If $\phi \in \mathcal{S}(X)_e$ then $W(\iota)\phi = \phi$.
If $\phi \in \mathcal{S}(X)_o$ then $W(\iota)\phi = -\phi$.
Then $\mathcal{S}(X)_e$ is the eigenspace for ι for the eigenvalue 1, and $\mathcal{S}(X)_e$ is the eigenspace, for $W(\iota)$ for the eigenvalue -1.

Since ι is central in Sp then $\mathcal{S}(X)_e$ and $\mathcal{S}(X)_o$ are each invariant under all $W(g)$, $g \in Sp$.

It can be shown that $\mathcal{S}(X)_e$ and $\mathcal{S}(X)_o$ give irreducible projective representations of $Sp(V)$.
Define

$$GSp(V) = \{\, g \in GL(V) : \langle gv, gw \rangle = d\,\langle v, w \rangle \,\}$$
$$\text{for some } d = d(g) \in K^*, \quad v, w \in V \,\}$$

There is an action of GSp on H by

$$g(c, v) = (dc, gv), \quad c \in K,\ v \in V.$$

Then ${}^gS$ is a Schrödinger representation of H, with central character $\lambda[d]$ where $\lambda[t](c) = tc$, $c \in K$. Take $t \in K^*$ and let $g = tI \in Gsp$, $d(g) = t^2$. Since g commutes with all $g' \in Sp(V)$, the Weil representation for ${}^gS$ is the same as the one for S.

Write $W = W_\lambda$. Then we see that

$$W_\lambda = W_{\lambda[t^2]}, \quad t \in K.$$

If we replace each $W(g)$ by a scalar multiple $c(g)W(g)$, then we get a new 2-cocycle, say $\beta(g_1, g_2)$, where

$$\beta(g_1, g_2) = c(g_1)c(g_2)\alpha(g_1, g_2)/c(g_1 g_2).$$

$$H^2(G, \mathbb{C}^*) = Z^2(G, \mathbb{C}^*)/B^2(G, \mathbb{C}^*)$$

where

$$\alpha \in Z^2(G, \mathbb{C}^*) : \alpha(gh, k)\alpha(g, h) = \alpha(g, hk)\alpha(h, k)$$

$g, h, k \in G$.

$$\alpha \in B^2(G, \mathbb{C}^*) : \alpha(g,h) = c(g)c(h)/c(gh)$$

for some $c : G \to \mathbb{C}^*$.

Then α and β are equal in $H^2(G, \mathbb{C}^*)$.
Suppose that K is finite.

If T is a projective representation of G of finite degree m, with 2-cocycle α, $T(g_1g_2) = \alpha(g_1, g_2)T(g_1)T(g_2)$ implies

$$\det T(g_1g_2) = \alpha(g_1, g_2)^m \det T(g_1) \det T(g_2).$$

So $\alpha^m = 1$ in $H^2(G, \mathbb{C}^*)$.

Suppose the operators $T(g)$ act on the vector space $\mathcal{V}$. Suppose that $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ and that $\mathcal{V}_1$ and $\mathcal{V}_2$ are each invariant under all $T(g)$.

Then we get projective representations T_1 and T_2, with the same cocycle α. If $m_2 = \dim \mathcal{V}_2 = \dim \mathcal{V}_1 + 1 = m_1 + 1$, then α^{m_1+1} and α^{m_1} are both trivial in H^2, and so is α.
In our case $\mathcal{V}_1 = \mathcal{S}(X)_o$, $\mathcal{V}_2 = \mathcal{S}(X)_e$.
In the real case, inside S is the Schwartz space of smooth functions on $\mathbb{R}^n$, tending rapidly to 0 at infinity. This gives another model of the Weil representation.

In the p-adic case, there is the Schwartz-Bruhat space $\mathcal{S}B$ of functions on K^n which are locally constant of compact support.

Suppose that K is a local field,

ring of integers R

maximal ideal $R\pi$

Recall $X = K^n$; let $L = R^n$.
Then L is compact, and so is $\pi^i L$ for any $i \in \mathbb{Z}$.
"Compact support" for $\phi \in \mathcal{S}B$ means that $\phi = 0$ outside $\pi^i L$ for some $i \in \mathbb{Z}$.

"Locally constant" means that ϕ takes the same values on all elements of the coset $\pi^i L + \pi^j L$ for some integer $j \geq i$.

For a given $\phi \in \mathcal{S}B$, we can adjust i and j so that $i = -j \leq 0$, $j \geq 0$; we can regard ϕ as a function on $\pi^{-j}L/\pi^j L$.

Let $\mathcal{S}_j$ denote the set of functions which are 0 outside $\pi^{-j}L$ and constant on cosets of π^{-j}/π^j.
For this page, assume that the conductor of λ, which is the largest fractional ideal of K on which $\lambda = 1$, is R.

Then $\mathcal{S}_j$ is invariant under the Weil operators $W(g)$ for $g \in Sp(2n, R)$.
Inside the vector space $V = K^{2n}$ we have the lattice $M = R^{2n}$; then $Sp(M) = Sp(R^{2n})$ is a maximal compact subgroup of $Sp(V)$. Restricted to $Sp(M)$, the Weil representation is a direct sum

$$W_0 \oplus \sum_{j=1}^{\infty} W_{2j}$$

where W_{2j} is a representation of $Sp(M/\pi^{2j}M)$ of dimension q^{2jn}.

The cocycle $\alpha = 1$ in $H^2(Sp(M/\pi^{2j}M), \mathbb{C}^*)$ just like the finite field case, so $\alpha = 1$ in $H^2(Sp(M), \mathbb{C}^*)$.
A subspace X of V is called Lagrangian if it is maximal totally isotropic (e.g. our X and Y above).
Given such X, then $V = X \oplus Y$ for some Lagrangian Y; $Y = X^*$.
One then can define the Schrödinger model $\mathcal{S}(X)$ as above.

Given two Lagrangians X_1, X_2, define

$$g_{X_2,X_1} : X_1/X_1 \cap X_2 \to (X_2/X_1 \cap X_2)^*, \quad (g_{X_2,X_1}(x_1), x_2) = \langle x_1, x_2 \rangle$$

Write $V = X_i \oplus Y_i$, $A_i = K \times Y_i$, $\mathcal{I}_i = \operatorname{ind}_{A_i}^H \lambda$

$$F_{X_2,X_1} : \operatorname{ind}_{A_1}^H \lambda \to \operatorname{ind}_{A_2}^H \lambda$$

$$F_{X_2,X_1} f(h) = \int_{X_2/X_1 \cap X_2} f(h(0, x_2)) |g_{X_1,X_2}|^{1/2} dx_2$$

Given $V = X \oplus Y$ as above, identify Y with X^*. If Q is a quadratic form on X, we get a map $s_Q : X \to X^*$ defined by Q. Define

$$L_Q = \{\, x + s_Q x : x \in X \,\} \subset V$$

It can be shown that L_Q is Lagrangian.

$$F_{X,L_Q} \circ F_{L_Q,Y} = \nu(Q) F_{X,Y}$$

$\nu(Q)$ is called the Weil index.
For $a \in K^*$ define $\nu(a)$ to be $\nu(Q_a)$ where $Q_a(x) = ax^2$.

In the finite case, $\nu(a) = \sum_{x\in K} ax^2$.
Weil shows that

$$\frac{\nu(ab)\nu(1)}{\nu(a)\nu(b)} = (a, b)$$

where (a, b) is the Hilbert symbol, which is 1 if a is a norm from $K(\sqrt{b})$.
$\nu(1)$ is an 8-th root of 1.
For Lagrangians X_1, X_2, X_3 define the quadratic form on $X_1 \oplus X_2 \oplus X_3$ by

$$Q(x_1, x_2, x_3) = \langle x_1, x_2\rangle + \langle x_2, x_3\rangle + \langle x_3, x_1\rangle \,.$$

Then Kashiwara's version of the Maslov (or Leray or Wall) index is the class of Q in the Witt group Witt_K.
One has

$$\nu(\tau(X_1, X_2, X_3))^2 = m(X_1, X_2)m(X_2, X_3)m(X_3, X_1)$$

$$m(X_1, X_2) = \nu(1)^{2(1-\dim X_1\cap X_2)}\nu(\det g_{X_1,X_2})^{-2}$$

Now take $g \in Sp(V)$, let $X_1 = X$, $X_2 = gX$. Let $s(g) = m(X_1, X_2) = m(X, gX)$.

As above, we have $\mathcal{I}_i = \mathrm{ind}_{A_i}^H \lambda$, $A_i = K \times Y_i$, $i = 1, 2$.
$Sp(V)$ acts on functions $f : H \to \mathbb{C}^*$ by $gf(h) = f(g^{-1}h)$. If $X_1 = X, X_2 = gX$, and if $f \in \mathcal{I}_1$ then $gf \in \mathcal{I}_2$.

Let $R(g)(f) = F_{X,gX}(gf)$. This is a "canonical" Weil representation. Its cocycle α satisfies

$$\alpha(g_1, g_2)^2 = s(g_1)^{-1}s(g_2)^{-1}s(g_1g_2).$$

So $\alpha^2 = 1$ in $H^2(Sp(V), \mathbb{C}^*)$.

That $\alpha \neq 1$ in $H^2(Sp(V), \mathbb{C}^*)$ follows from the non-triviality of the Hilbert symbol.
So we can replace $W(g)$ by $s(g)W(g)$, and then the resulting cocycle has values ± 1.

Define $Mp(V) = \{\, (\epsilon, g) : \epsilon = \pm 1,\ g \in Sp(V)\,\}$ called the metaplectic group:

$$(\epsilon, g)(\epsilon', g') = (\epsilon\epsilon'\alpha(g, g'), gg')$$

Define the Weil representation of $Mp(V)$ by

$$\widetilde{W}(\epsilon, g) = \epsilon W(g)$$

Then $\widetilde{W}$ is a true (linear, not projective) representation of $Mp(V)$.
Suppose that G_1 and G_2 are reductive subgroups of $Sp(2n, K)$, such that each is the centralizer in Sp of the other.

e.g. $V = V_1 \otimes_K V_2$
V_1 has a non-degenerate symmetric bilinear form, and
V_2 has a non-degenerate alternating bilinear form.
Then $V_1 \otimes V_2$ has a non-degenerate alternating bilinear form.

V_1 has isometry group $O(n_1, K) = G_1 \subset Sp(V)$
V_2 has isometry group $Sp(n_2, K) = G_2 \subset Sp(V)$
For $i = 1, 2$, let $\widetilde{G}_i$ be the inverse image in $Mp(V)$ of G_i.
The Weil representation $\widetilde{W}$ of $Mp(V)$ can be restricted to $\widetilde{G}_1 \times \widetilde{G}_2$.
Let π_1 be a smooth irreducible representation of $\widetilde{G}_1$. Let $A(\pi_1)$ be the maximal quotient of $\mathcal{S}(X)$ on which $\widetilde{W}(\widetilde{G}_1)$ acts as a multiple of π_1.

Then $\widetilde{W}(\widetilde{G}_2)$ acts on $A(\pi_1)$, so we have a representation of $\widetilde{G}_1 \times \widetilde{G}_2$, acting on $A(\pi_1)$.

A theorem of Waldspurger, confirming a conjecture of Howe, is that if the residue characteristic $p \neq 2$, then $A(\pi_1)$ is equivalent to $\pi_1 \otimes \pi_2$ for a unique smooth irreducible representation π_2 of $\widetilde{G}_2$. The representation π_2 is called the theta lift, or the Howe correspondent, of π_1.

More references

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