Hilbert-Siegel modular forms and representations

General:

F number field, $\hat{F}$ finite adeles

G / F

$$G_0 := G(\hat{F}), \quad G_\infty := \prod_{v \in S_\infty} G(F_v)$$

$$\begin{array}{ccc} G_0 & & G_\infty \\ \cup & & \cup \\ K_0 & & K_\infty \end{array}$$

fixed / choice here.

$$\mathcal{H}_0 = \mathcal{H}(G_0, \omega_0) \qquad \mathcal{H}_\infty = \mathcal{H}(G_\infty, K_\infty, \omega_\infty)$$

$$\mathcal{H} := \mathcal{H}_0 \otimes \mathcal{H}_\infty$$

Hilbert space

$$L^2 := L^2(G(F)\backslash G(\mathbb{A}), \omega)$$

$\mathcal{H}$-algebra

$$\mathcal{A}(G, \overset{\text{type}}{J}) = \{\phi : G(\mathbb{A}) \to \mathbb{C} \mid \cdots\} \qquad J \triangleleft \mathfrak{z}$$

af1. $\underset{\sim\sim\sim}{\text{smooth}}$, slow growth

af2. $\forall X \in J$, $X\phi = 0$

af3. $\phi(z\gamma g) = \omega(z)\,\phi(g)$

$\forall \gamma \in G(F)$, $g \in G(\mathbb{A})$

$\forall z \in Z(\mathbb{A}_F)$

$\mathcal{H}$ elementary idempotents:

$U \subset G_0$ c.o.s.

$e_U \in \mathcal{H}_0$ idemp.

$\sigma : K_\infty \xrightarrow{\text{irrep}} \mathrm{Aut}_{\mathbb{C}}(W)$ f.d.

$e_\sigma \in \mathcal{H}_\infty$ idemp

Elementary idemp.s

$\xi \in \mathcal{H}$, $\xi = e_U \otimes e_\sigma$

level

weight

af4: $\exists\ \xi$ (elem. idempt), $\underline{R(\xi)\phi = \phi}$

$$(R(\xi)\phi)(x) = \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} \xi(y)\,\phi(xy)\,dy$$

af5. cuspidality.

<u>Prop'n</u> $\mathcal{A}(G,J)$ is a Hecke-alg. in L^2
automorphic forms, type J

Recall: $A(G,\xi,J) = \mathcal{A}(G,J)^{\xi}$

$\mathcal{H}(\xi)$-alg.

$J \triangleleft \mathfrak{z} \subseteq$ centre of univ. alg. of

$\mathfrak{g}_\infty \otimes_{\mathbb{R}} \mathbb{C}$

G_∞ (real) Lie group | $G_\infty = \prod_{v \in S_\infty} G(F_v)$

$\mathfrak{g}_\infty = \mathrm{Lie}\, G_\infty$

$$(X\phi)(x) = \frac{d}{dt}\phi(x \exp tX)\Big|_{t=0}$$

$\mathcal{A}(G, J)$ is a $\mathcal{H}$-alg.

a$\underline{f\ 1}$. ϕ smooth, slowly increasing

$\forall c \in G_0$, $\phi_c : G_\infty \longrightarrow \mathbb{C}$ smooth

where $\phi_c(x) = \phi(\underbrace{\underset{G_0}{c}\ \underset{G_\infty}{x}}_{(x \in G(A))})$

Exercise

$$f \in \mathcal{H},\ \phi \in \mathcal{A}_0(G,J) \Rightarrow \underline{R(f)}\phi \in \mathcal{A}_0(G,J)$$

Vector-valued auto. forms:

$$\mathcal{A}(G, \mathcal{U}, W, J) = \{ \Phi : G(\mathbb{A}) \longrightarrow W \mid \ldots \}$$

$\mathcal{U}$ c.o.s in G_0

$\delta : K_\infty \xrightarrow[\text{unitary}]{\text{irrep}} \mathrm{Aut}_{\mathbb{C}}(W)$

$\mathcal{U}, \delta \rightsquigarrow e_{\mathcal{U}} \otimes e_\delta =: \mathfrak{Z}$

$\mathcal{A}(G, \mathfrak{Z}, J)$ ($\mathcal{A}_0$, AF5. cuspidality) $\mathcal{H}(\mathfrak{Z})$-alg.

AF1. Φ smooth, slow

AF2. $J\Phi = 0$ $k \in \mathcal{U}$

AF3. $\Phi(z\gamma g k) = \omega(z)\Phi(g)$

AF4. $\Phi(g k_\infty) = \delta(k_\infty^{-1})\Phi(g)$

Prop: $\mathcal{A}(G, \mathcal{U}, W, J) \cong \mathcal{A}(G, \mathfrak{Z}, J)$

Prop'n $\mathcal{A}(G, U, W, J)$ is a $\mathcal{H}(\xi)$-alg. iso. to $\mathcal{A}(G, \xi, J)$

Proof $f \in \mathcal{H}(\xi)$, $\Phi \in \mathcal{A}(G, U, W, J)$

$$(R(f)\Phi)(z\gamma gk)$$

$$= \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} f(y)\, \Phi(z\gamma g k y)\, dy$$

$$= \omega(z) \int f(y)\, \Phi(gky)\, dy$$

$$= \omega(z) \int f(y)\, \Phi(gky)\, dy$$

$$ky = xk$$

$$= \omega(z) \int f(k^{-1}xk)\, \Phi(gxk)\, dx$$

$$y = k^{-1}xk$$

$$dy \overset{!}{=} dx$$

$$= \omega(z) \int f(k^{-1}xk)\, \Phi(gx)\, dx$$

$$k \in U$$

$$f \in \mathcal{H}(\mathfrak{z}) \Rightarrow f(k^{-1}xk) = f(x)$$

$$= \omega(z) \big(R(f)\phi\big)(g)$$

$$\mathcal{A}(G, U, W, J) \longrightarrow \mathcal{A}(G, \mathfrak{z}, J)$$

$$\Phi \longmapsto \Phi_w \qquad w \neq 0$$

$$\Phi_w(g) = \langle \Phi(g), w \rangle$$

Clear that this is linear & Injective:

$\Phi_w = 0$ in $\mathcal{A}(G, \mathfrak{z}, J)$

$\Rightarrow \quad \langle \Phi(g), w \rangle = 0, \ \forall g \in G(\mathbb{A})$

$\Rightarrow \forall g, \forall k_\infty \in K_\infty, \ \langle \Phi(g k_\infty), w \rangle = 0$

$\Rightarrow \ \ldots \ \langle \sigma(k_\infty^{-1}) \Phi(g), w \rangle = 0$

$\Rightarrow \forall g, k_\infty, \quad \langle \Phi(g), \sigma(k_\infty) w \rangle = 0$

$\Rightarrow \forall g, \Phi(g) \in (\underbrace{K_\infty \cdot w})^\perp = W^\perp$

$\Rightarrow \forall g, \Phi(g) = 0$

$\Rightarrow \Phi = 0$. in $\mathcal{A}(G, U, W, J)$.

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§0 : Review

§1 : Vector-valued ....

§2 : $GSp(2n)$